Dynamic Control of Multiclass Omnichannel Hybrid Production Systems: Applications to Dynamic Pricing in Restaurants

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Abstract

The rise of omnichannel production systems has transformed various industries, introducing new challenges in demand and operations management. One notable development is the adoption of dynamic pricing in omnichannel production systems, as evidenced by recent developments in the restaurant industry. Motivated by these trends, we develop a stochastic processing network model for a multiclass omnichannel hybrid production system serving heterogeneous, price- and delay-sensitive customers. The firm offers a mix of make-to-order (MTO) and make-to-stock (MTS) goods, available through both walk-in and online channels, at multiple quote times. MTO orders incur earliness and tardiness costs, while MTS orders incur tardiness and inventory holding costs. Walk-in customers are impatient and may abandon if their waiting time is excessive. The objective is to maximize the long-run average expected profit through dynamic pricing, production scheduling, and order rejection decisions. Since this problem appears analytically intractable, we consider an approximating Brownian control problem in the heavy-traffic regime. We prove that the optimal policy is a two-sided barrier policy with a state-dependent drift rate, for which we derive a closed-form solution. We then propose a joint dynamic pricing, scheduling, and rejection policy by interpreting this solution in the context of the original production system. Finally, we demonstrate the effectiveness of our proposed policy through a comprehensive simulation study and offer several managerial insights. For instance, we show that the value of dynamic pricing is greatest when demand is predominantly online or MTS, and that dynamically pricing the online channel alone—while keeping walk-in prices fixed—can yield substantial gains, making it a natural entry point for implementation.

Keywords: omnichannel; restaurant; dynamic pricing; quote time; scheduling; abandonment; make-to-order; make-to-stock; stochastic processing network; heavy traffic analysis; stochastic control

1 Introduction

Omnichannel systems, which integrate multiple demand channels, have become increasingly prevalent across many industries, such as retail, healthcare, and hospitality. Production systems have also embraced omnichannel practices, giving rise to omnichannel production, in which the production and fulfillment processes are dynamically coordinated across multiple channels. The restaurant industry exemplifies this trend, as restaurants must simultaneously manage both walk-in and online orders. Online sales now account for a substantial portion of restaurant revenue. In 2023, U.S. restaurant sales surpassed $1 trillion for the first time, with online orders accounting for 23% of total sales (National Restaurant Association, 2024; TouchBistro, 2024b). Starbucks, for instance, reports that mobile orders now account for over 30% of all U.S. sales (Soper, 2024). This trend continues to accelerate, with 25% of restaurants reporting a significant increase in online sales over the past year (TouchBistro, 2025).

The integration of multiple customer channels enhances flexibility and expands market reach, but it also introduces significant operational challenges for production systems. Many such systems, including restaurants, must coordinate fulfillment across both walk-in and online channels while managing the production of make-to-order (MTO) and make-to-stock (MTS) goods. MTO goods, such as customizable meals and beverages, can be tailored to customer specifications and are produced only after an order is received. Their preparation must be carefully timed, particularly for online MTO orders: if prepared too early, they incur earliness costs, as food degrades in quality before the customer arrives—leading, for example, to "lukewarm lattes" or "cold burritos" (Ghosh et al., 2023). According to a recent industry report, receiving cold food is a leading complaint, with 20% of diners citing it as their biggest frustration (TouchBistro, 2024a). In contrast, MTS goods, such as premade drinks and salads, can be prepared in advance and stocked for immediate sale, but they incur holding costs. Regardless of product type, tardiness costs arise when orders are delayed—a common source of customer dissatisfaction, with about 12% of diners citing food not being ready upon arrival as their biggest frustration (TouchBistro, 2024a). Walk-in customers are particularly impatient: if waiting times are too long, they may abandon their purchase entirely. More than half of diners indicate they would leave the restaurant if their wait exceeds 30 minutes (TouchBistro, 2024a).

To address these challenges, omnichannel production systems are increasingly adopting automation and intelligent pricing strategies to optimize their operations. This is particularly important in the restaurant industry, given its traditionally tight profit margins. Many restaurants have already integrated automation into key aspects of their operations, including online ordering, staff scheduling, and inventory management (TouchBistro, 2025). For example, Starbucks is improving its operations to address persistently long waiting times. As reported by *The Wall Street Journal*, the company has developed a rules-based order scheduling algorithm to replace its previous first-come, first-served fulfillment strategy (Haddon and Bousquette, 2025); see also Rogers (2024) and QSR Magazine (2025). The algorithm has already reduced waiting times by two minutes at pilot locations. Starbucks is also experimenting with scheduling specific pickup times for mobile orders and training baristas to time order preparation based on factors like order complexity, aiming to prevent drinks from sitting out before the customer arrives.

As restaurants seek more ways to manage congestion and reduce waiting times, one emerging approach is dynamic pricing. The increasing adoption of digital menus and online ordering platforms has made it easier to adjust prices dynamically in response to congestion.[1] Although still in its early stages, dynamic pricing has already shown measurable success in the restaurant industry, with multiple startups, including Chowly, DynamEat, Juicer, Priceff, and Sauce, developing dynamic pricing solutions tailored to restaurants. These strategies are already being implemented. For example, Sauce has helped Piada Italian Street Food increase profits and reduce waiting times (Sauce Technologies, 2025). Other major restaurant chains, such as Koti Pizza (Finland's largest pizza chain) and Noodles & Company (a U.S.-based pasta chain), have also reported success with dynamic pricing (Hynum, 2021; Clyne-Canham, 2023b). These examples reflect a broader trend, with many other restaurants adopting congestion pricing and reporting success.

Motivated by these recent industry trends, this paper develops a stochastic processing network model for a multiclass omnichannel hybrid production system that serves heterogeneous, price- and delay-sensitive customers. The firm offers a mix of MTO and MTS goods through walk-in and online channels, with online customers selecting from multiple predetermined quote times. The firm seeks to maximize the long-run average expected profit by jointly making dynamic pricing, scheduling, and order rejection (i.e., admission control) decisions. Scheduling decisions consist of two types of activities: production and reallocation. Production activities involve manufacturing goods, while reallocation activities redirect available MTS inventory to satisfy online orders. Reallocation activities are needed because, unlike MTO goods, which must be produced specifically for each order, MTS goods are inherently flexible in that they can be used to satisfy demand from either channel. MTO orders incur earliness and tardiness costs, while MTS orders incur tardiness and holding costs. Finally, walk-in customers are impatient and may abandon the system if waiting times are excessive, resulting in abandonment costs.

Because the joint dynamic pricing, scheduling, and rejection control problem is analytically and computationally intractable, this paper adopts the methodology proposed by Harrison (1988, 2000, 2003) and analyzes the system in the heavy-traffic regime, where the system capacity is nearly fully utilized. In this regime, the original system can be approximated by a diffusion control problem, known as the Brownian control problem. We prove that the Brownian control problem is equivalent to a one-dimensional drift rate control problem, whose state process—referred to as the workload process—represents the total backlog in the system measured in hours of total work. We solve this drift rate control problem by solving the associated Bellman equation and establish that the optimal policy is a two-sided barrier policy with a state-dependent drift rate, for which we derive a closed-form solution. The optimal policy comprises a drift rate control policy, a workload configuration policy, and a two-sided barrier policy. The rejection and idling processes maintain the workload process between the lower and upper barriers, respectively. Between the two barriers, the drift rate and workload configuration policies control the workload, which are ultimately interpreted as dynamic pricing and scheduling policies for the original production system.

This paper extends the existing literature by making several key modeling contributions. First, we

[1] While congestion pricing is relatively new to the restaurant industry, time-of-day pricing strategies have existed for decades, including happy hours and early bird specials.

consider a multiclass hybrid production system, thereby extending earlier models that exclusively focus on either MTO or MTS goods. Research on hybrid production systems remains relatively limited, and this paper contributes to the understanding of their operational complexities. Second, we consider an omnichannel production system where each good is offered at multiple quote times, thereby extending the literature (see, e.g., Farahani et al. (2022) and Gao et al. (2023a)) which focuses on a single MTO good offered at a single quote time. A distinguishing feature of our model is that, since MTS goods can be produced in advance, their inventory can be used to fulfill demand across all channels requesting the same good. Consequently, the presence of online MTS goods necessitates a more complex modeling framework than conventional queueing systems, requiring a stochastic processing network model that accommodates both production and reallocation activities. Third, to the best of our knowledge, this is the first paper to study dynamic pricing in omnichannel or hybrid production systems—let alone in an integrated omnichannel hybrid setting. Notably, although the importance of dynamic pricing in the restaurant industry has long been recognized (see, e.g., Kimes and Beard (2013) and Roy et al. (2022)), research on this topic remains limited. Finally, we incorporate customer abandonment, which fundamentally impacts both the analysis and the structure of our proposed policy. In the presence of customer abandonment, the system manager cannot rely on greedy workload distribution policies based solely on immediate costs; instead, the optimal configuration crucially depends on the workload level, state costs, and the value function.

This paper makes several technical contributions and reveals structural insights. From a technical perspective, we solve a drift rate control problem with convex piecewise linear state costs, endogenously determined reflecting barriers, and a workload configuration that depends jointly on the system workload and the value function. A key challenge in this analysis arises from the distinct cost structure induced by the interplay between the online channel, MTO and MTS goods, and customer abandonment. This leads to a Bellman equation whose structure varies across workload levels, requiring a careful treatment of several technical subtleties to derive the solution explicitly. Our analysis also yields several structural insights. First, the pricing, rejection, and scheduling decisions depend primarily on the aggregate workload, which evolves on a slower time scale than the underlying demand and production processes; as a result, prices change slowly over time. Second, in large systems, the magnitude of the prescribed price adjustments is small, which aligns with customer and regulatory preferences (Clyne-Canham, 2023a). Third, as the backlog of work increases and inventory levels decrease, prices should be adjusted to decrease the effective demand rate, and scheduling decisions should be adjusted to maintain a low effective state cost (comprising earliness, tardiness, holding, and abandonment costs). Finally, order rejection is worthwhile only when the backlog of work is excessive, and it is sufficient to reject orders from a single class.

The simulation study offers several key managerial insights into when dynamic pricing delivers the greatest value in omnichannel hybrid production systems. First, the value of dynamic pricing (relative to static pricing) increases with the system load, making it particularly effective for heavily loaded systems. Second, the value of dynamic pricing decreases with the abandonment rate; that is, abandonments serve as a partial substitute for dynamic pricing by reducing the effective server utilization. Third, the value of dynamic pricing increases with the share of online demand. As such, businesses with substantial online demand—or the ability to shift demand online—stand to gain the most from dynamic pricing. A similar pattern, albeit to a lesser extent, holds for MTS demand, suggesting that businesses with significant MTS demand are also well-positioned to benefit from dynamic pricing. Finally, for businesses with a non-negligible share of online demand, dynamically pricing the online channel alone yields substantial gains. Focusing first on the online channel thus offers a natural entry point for adopting dynamic pricing before broader deployment.

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the model and problem formulation. Section 4 introduces a Brownian approximation of the original control problem, and Section 5 reformulates it as a lower-dimensional, more tractable stochastic control problem known as the equivalent workload formulation. Section 6 solves the equivalent workload formulation. Section 7 proposes a solution to the original control problem using the solution to the equivalent workload formulation. Section 8 evaluates the effectiveness of the proposed policy through an extensive simulation study and offers several managerial insights. Several appendices supplement the main body. Appendix A provides a formal derivation of the Brownian control problem introduced in Section 4. Appendix B solves the Bellman equation introduced in Section 6. Appendix C contains the proofs of the main technical results from Sections 4–6. Appendix D provides supplementary material for the simulation study in Section 8.

2 Literature Review

This paper relates to four main streams of literature: (i) scheduling of production systems, (ii) control of queueing systems, (iii) dynamic pricing of production and queueing systems, and (iv) control of omnichannel systems. The first stream of literature studies scheduling in production systems, a well-established area of research; see Nahmias and Olsen (2015, Chapter 9) for an overview. Most of the literature has focused on either MTS or MTO production systems. The earliest work on MTS production scheduling is Wein (1992), which proposes a dynamic scheduling policy for a multiclass MTS system using heavy-traffic analysis. Subsequent notable contributions on scheduling of MTS production systems include Veatch and Wein (1996), Perez and Zipkin (1997), de Véricourt et al. (2000), Xu and Chao (2009), and Özkan and Tan (2025); see Ata and Barjesteh (2022) for an overview. Similarly, research on MTO production systems is extensive, with some notable papers including Plambeck (2004), Ata (2006), Ata and Olsen (2009), Rubino and Ata (2009), Afèche and Pavlin (2016), Chen et al. (2022b), and Sun and Zhu (2024).

Despite their widespread industry prevalence, hybrid production systems, which combine MTS and MTO production, have received considerably less attention; see Peeters and van Ooijen (2020) for an overview. An early paper in this area is Carr and Duenyas (2000), which considers a two-product hybrid production system where MTS demand must be met through available inventory or an external supplier at a cost, while the MTO demand can be rejected. The system manager makes scheduling decisions for both products and admission control decisions for the MTO product, and shows that the optimal policy is characterized by a switching curve. Iravani et al. (2012) extends Carr and Duenyas (2000) by allowing backlogging for MTS orders, and shows similar structural properties. The closest paper to ours in this stream is Markowitz and Wein (2001), which studies the scheduling of a multiclass hybrid production system with switching costs in heavy traffic. While it shares certain features with ours, such as quote times and earliness/tardiness costs, it focuses on dynamic cyclic policies due to switching costs and relies on computational methods. In contrast, our paper assumes zero switching costs, incorporates admission control and pricing decisions, and derives a closed-form solution to the diffusion control problem.

The second stream of literature studies the control of queueing systems; see Stidham (2002) for an overview. From a methodological perspective, our paper builds on the literature of dynamic control of queueing systems in heavy traffic, pioneered by Harrison (1988, 2000, 2003). The standard approach is to approximate the original queueing system with a Brownian system, resulting in a more tractable formulation. Early examples of this approach include Harrison and Wein (1989), which studies optimal sequencing for a crisscross network, and Harrison and Wein (1990), which studies a multiclass two-station closed queueing network. Since then, this approach has been used to study a wide range of queueing problems; see, e.g., Plambeck et al. (2001), Plambeck (2004), Ata and Olsen (2009), Ata and Tongarlak (2013), Ghamami and Ward (2013), Kim et al. (2018), and Özkan and van Houtum (2023).

In the heavy-traffic literature, pricing and service rate control problems are typically approximated by drift rate control problems for the Brownian system. A seminal paper in this area is Ata et al. (2005), which studies a drift rate control problem of a reflected Brownian motion on a bounded interval. Ghosh and Weerasinghe (2007) extends Ata et al. (2005) by incorporating nondecreasing holding costs with an endogenously chosen upper barrier. Several papers have studied applications of drift rate control; see, e.g., Ata (2006), Ghosh and Weerasinghe (2010), Budhiraja et al. (2011), Ata and Tongarlak (2013), Weerasinghe (2015), Ata et al. (2019), Ata and Barjesteh (2022), Alwan et al. (2024), and Ata et al. (2024).

Dynamic control of queueing systems with abandonments in heavy traffic has also received some attention. Much of this literature assumes exponential patience times, as we do, which results in a linear drift rate in the Brownian control problem; see, e.g., Ward and Kumar (2008), Ghosh and Weerasinghe (2010), Ata and Tongarlak (2013), Ghamami and Ward (2013), and Ibrahim et al. (2017). For extensions that consider non-exponential patience times, see Reed and Ward (2008), Ibrahim and Whitt (2009), Bassamboo and Randhawa (2015), and Kim et al. (2018). Our paper builds on the abandonment literature by studying an omnichannel hybrid production system and incorporating pricing. We show that in such systems, abandonments do not impact the workload process when it is below a threshold dictated by the server utilization and quote times of the online classes. However, above this threshold, abandonments generate a negative drift rate in the workload process. Also, pricing and rejection decisions crucially depend on abandonment rates and costs.

The third stream of literature studies dynamic pricing of production and queueing systems; see Gallego and Topaloglu (2019) for an overview. Some studies adopt a dynamic programming approach (Ata and Shneorson, 2006), some use greedy heuristics (Wu et al., 2024) and others, including our work, employ diffusion approximations (Ata and Olsen, 2009; Kim and Randhawa, 2017; Alwan et al., 2024). The most related papers are Çelik and Maglaras (2008) and Ata and Barjesteh (2022). Çelik and Maglaras (2008) studies a multiclass MTO production system with multiple predetermined lead times, where the system manager makes dynamic pricing, lead-time quotation, scheduling, and expediting control decisions to maximize the long-run average expected profit. We build on this paper by incorporating MTS products, the walk-in channel, state (earliness, tardiness, and holding) costs, and customer abandonments. Ata and Barjesteh (2022) studies a multiclass MTS production system with linear holding and backlogging costs, where the system manager makes dynamic pricing, outsourcing, and scheduling decisions to maximize the long-run average expected profit. Gao and Huang (2023) extends this paper by incorporating strictly convex state costs and proving asymptotic optimality. We build on Ata and Barjesteh (2022) by incorporating MTO products, the online channel (with its unique cost structure), and customer abandonments. Our problem leads to a novel workload formulation, in which the state cost function is a non-V-shaped, piecewise linear function that depends on the value function. Our scheduling policy is nongreedy (due to abandonments) and may idle even with a positive workload (due to the online classes). Our pricing policy also crucially depends on the quote times, state costs, abandonment rates, and abandonment costs.

The literature on lead-time quotation typically imposes rigid due-date constraints, as in Plambeck (2004), Çelik and Maglaras (2008), and Afèche (2013). However, in many applications—such as restaurants—orders may be fulfilled before or after the quoted time, incurring earliness or tardiness costs, respectively. The literature explicitly modeling earliness and tardiness costs remains limited. Early papers, such as Pandelis and Teneketzis (1994) and Righter (1996), restrict to non-idling scheduling policies. But, as Markowitz and Wein (2001) and Farahani et al. (2022) correctly note, idling is a critical component in systems with earliness costs. Accordingly, we do not impose a non-idling constraint. Farahani et al. (2022) considers an online food-ordering service with a single class of customers and a single quote time. The objective is to minimize the long-run average expected earliness and tardiness costs by making scheduling decisions. The paper proposes a family of static threshold policies, where the server operates only when the time remaining until the pick-up time of the head-of-line customer drops below a threshold. They prove asymptotic optimality of static threshold policies in several regimes. An closely related paper to ours is Gao et al. (2023a), which extends this line of work by introducing a walk-in class, thereby giving rise a two-class omnichannel service system. The objective is to minimize the long-run average expected cost by making dynamic scheduling and order rejection decisions. We build on Gao et al. (2023a) by incorporating MTS products, multiple goods, multiple quote times, abandonments, and dynamic pricing. These features give rise to a fundamentally different workload formulation, for which we propose a closed-form solution.

The fourth stream of literature studies the control of omnichannel systems, with most research focusing on omnichannel retail; see Hübner et al. (2022) for an overview. In omnichannel retail, goods are ordered in advance and inventoried to meet demand across multiple channels. In contrast, omnichannel production, which is the focus of this paper, involves producing goods using limited server capacity to meet demand. Within omnichannel retail, some explored topics include the impact of self-order technologies (Gao and Su, 2018), information sharing (Roet-Green and Yuan, 2020; Baron et al., 2023; Roet-Green and Yang, 2024), pricing decisions (Harsha et al., 2019), return policy decisions (Nageswaran et al., 2020), and fulfillment for online orders (Gao et al., 2023b; Baron et al., 2024; Guo et al., 2024). Closer to our setting is the emerging literature on omnichannel service systems; see, e.g., Gao and Su (2018), Chen et al. (2022a), Gao et al. (2023a), Ghosh et al. (2023), and Kang et al. (2024). A production system composed solely of MTO products resembles a service system, with production occurring only in response to realized demand; however, MTS products can be produced in advance and stored in inventory to meet future demand. We build on this literature by incorporating MTS goods, multiple goods and quote times, customer abandonments, and dynamic pricing decisions.

3 Model

We consider a production system that serves a market of heterogeneous, price- and delay-sensitive customers. The system is hybrid in that it produces both make-to-order (MTO) and make-to-stock (MTS) goods. Moreover, it is omnichannel in that the goods are available through two channels: walk-in and online. MTO goods are produced only after an order is received, while MTS goods may be pre-produced and held as (finished goods) inventory. Accordingly, MTO goods may be viewed as "customized," and MTS goods as "standardized." Each good can be offered through one or both channels. Goods offered through the online channel are available at multiple quote times, defined as the time between when the order is placed and the promised pick-up time. Each online good may be offered at a different set of quote times. Although goods ordered through the walk-in channel do not have an associated quote time, for modeling convenience, we assign them a quote time of zero. The menu of offered goods, the MTO/MTS designation of each good, and the available quote times for each good are assumed to be predetermined and thus fixed throughout.

We define a "product" as a combination of a good and a quote time; see Çelik and Maglaras (2008) for a similar treatment. We numerically label each product and let $\mathcal{S} := \{1, \dots, K\}$ denote the set of all products, where $K \in \mathbb{N} := \{1, 2, \dots\}$ denotes the total number of distinct products. Product $k \in \mathcal{S}$ has a quote time of $\delta_k \in \mathbb{R}$, where $\delta_k > 0$ for online products and $\delta_k = 0$ for walk-in products. We assume that class k customers, i.e., customers ordering product k, arrive to pick up their goods precisely δ_k time units after placing their order. For concreteness, consider an illustrative example with one MTO good and one MTS good, each offered through the walk-in channel and at two quote times (15 and 30 minutes) in the online channel. Figure 1 provides an illustration of our model for this $K = 6$ product example. The modeling components shown in the figure will be introduced and explained in the sections that follow.

For ease of exposition, we partition the set of products $\mathcal{S}$ into the following four subsets: walk-in MTO products $\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}$, online MTO products $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}$, walk-in MTS products $\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$, and online MTS products $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$. We also let

$$\mathcal{S}^{\mathrm{MTO}} \coloneqq \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}, \quad \mathcal{S}^{\mathrm{MTS}} \coloneqq \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \quad \mathcal{S}_{\mathrm{w}} \coloneqq \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \quad \mathcal{S}_{\mathrm{o}} \coloneqq \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$$

denote the set of MTO products, MTS products, walk-in products, and online products, respectively. We assume that at least one MTS product is offered (i.e., $\mathcal{S}^{\mathrm{MTS}} \neq \varnothing$) and that all MTS goods are available through the walk-in channel.[2] For $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$, let $w(k) \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ denote the corresponding walk-in MTS product that shares the same good as (online MTS) product k. Moreover, for $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$, we define $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k) \coloneqq \{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}} : w(j) = k\}$ as the set of online MTS products that share the same good as (walk-in MTS) product k.

The system manager controls the system by making dynamic pricing, scheduling, and order rejection decisions. Section 3.1 discusses the demand structure and pricing decisions. Section 3.2 introduces our stochastic processing network model, and discusses the scheduling and rejection decisions. Section 3.3 discusses the state dynamics. Finally, Section 3.4 discusses the cost structure and the stochastic control problem.

3.1 Demand Structure

The production system operates in a market of price-sensitive customers. Customers arrive according to a nonhomogeneous Poisson process, whose intensity depends on the prices set by the system manager. To formalize this, let $\mathcal{P} \subset \mathbb{R}_+^K$ denote the set of admissible price vectors. The system manager chooses a price vector $p(t) = (p_1(t), \ldots, p_K(t)) \in \mathcal{P}$ at each time t, where $p_k(t)$ denotes the price of product k at time t. The corresponding demand rates are determined by a demand function $\Lambda : \mathcal{P} \to \mathbb{R}_+^K$ that maps the price vector to an instantaneous demand rate vector. The instantaneous demand rate vector at time t is denoted by $\lambda(t) = \Lambda(p(t))$, where $\lambda_k(t)$ denotes the instantaneous demand rate of product k at time t. The set of admissible instantaneous demand rate vectors is given by $\mathcal{L} \coloneqq \{\Lambda(p) : p \in \mathcal{P}\} \subseteq \mathbb{R}_+^K$. We refer to $p = \{p(t) : t \geq 0\}$ as the price process and to $\lambda = \{\lambda(t) : t \geq 0\}$ as the instantaneous demand rate process. To ensure analytical tractability, we impose the following regularity assumption on the demand function:

Assumption 1. *There exists an inverse demand function $\Lambda^{-1} : \mathcal{L} \to \mathcal{P}$ that maps each achievable instantaneous demand rate vector to the corresponding price vector that induces it.*

This assumption is satisfied by many common demand functions, such as linear, exponential, and logit

[2] If no MTS good is offered, the Bellman equation reduces to one with a fixed lower barrier and would require some slight modifications to the analysis. Similarly, the assumption that all MTS goods are available through the walk-in channel could be relaxed to allow for online-only MTS goods, but this would introduce notational complexities without yielding significant insights.

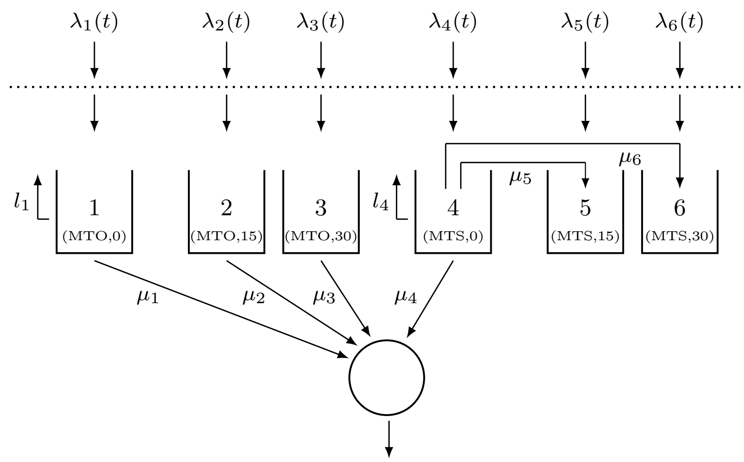

Figure 1: An illustration of the stochastic processing network model for our six-product illustrative example. Each buffer is labeled by its product class, along with its MTO/MTS designation and quote time. Products 1 and 4 belong to the walk-in channel, while products 2, 3, 5, and 6 belong to the online channel. There are four production activities and two reallocation activities. The dotted line denotes the rejection decisions. The upward arrows denote abandonments.

demand models; see, e.g., Talluri and Van Ryzin (2006) and Çelik and Maglaras (2008). This assumption allows us to treat the instantaneous demand process as the system manager's control, since the corresponding price process can be recovered using the inverse demand function. Let $\gamma_k > 0$ denote the variable cost of production for product $k \in \mathcal{S}$.[3] We next define the instantaneous profit rate function $\Pi : \mathcal{L} \to \mathbb{R}$ as follows:

$$\Pi(x) \;:=\; x'\big(\Lambda^{-1}(x) \;-\; \gamma\big), \quad x \in \mathcal{L}, \tag{1}$$

where $\gamma = (\gamma_1, \ldots, \gamma_K)$. This function represents the revenue rate net of variable cost of production in an idealized setting absent congestion-related concerns (e.g., in a system without stochastic variability) assuming that all orders are accepted and successfully fulfilled.

3.2 Stochastic Processing Network Formulation

We model the production system as a single-server stochastic processing network, following the widely adopted framework introduced in Harrison (2000, 2003); see Figure 1 for an illustration. Stochastic processing networks generalize queueing networks, see, e.g., Harrison (1988), and consist of three basic elements: jobs, servers, and activities. The notion of jobs and servers is identical to that of a queueing network. In our setting, jobs represent product orders and the server represents the production system. Each product class has a dedicated buffer where the orders of that class are stored. However, the distinctive feature of a stochastic processing network is that an "activity" can accommodate more complex operational tasks in order to fulfill orders. (That is, an activity is not limited to a single server serving a single buffer.) In

[3] In our model, MTS goods are initially stored in their walk-in buffers and later used by reallocation activities to satisfy online MTS demand (see Section 3.2). The inclusion of the variable production cost for the online MTS goods in the profit rate function is thus a modeling convention to maintain consistency.

our setting, activities correspond to the system manager's scheduling decisions and fall into two distinct categories—production and reallocation—which we describe next.

Production Activities. Production activities involve the production of MTO products and walk-in MTS products: for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$, activity k produces one unit of the good associated with product k. Note that online MTS products do not have dedicated production activities; their associated goods are produced via the production activity of the corresponding walk-in MTS product and can be later used to fulfill online orders using reallocation activities, as we will discuss next. This modeling structure provides the system manager with increased operational flexibility, enabling dynamic allocation of MTS goods rather than committing to a specific product at the time of production.

The duration of production activity k, i.e., the production time of the good associated with product k, follows a general distribution with mean $m_k > 0$ and coefficient of variation $c_{sk} > 0$. We let $\mu_k \coloneqq 1/m_k$ for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$ denote the production rate for product k. Let $S_k(t)$ denote the total number of class k products produced until time t if the system were to continuously work on class k products up to time t. Since online MTS products are fulfilled using the inventory of the corresponding walk-in MTS products, we define the mean production time of an online product to be equal to that of its associated walk-in product. As such, we define the K-dimensional mean production time vector $m = (m_1, \ldots, m_K)$ as follows:

$$m_k \coloneqq \begin{cases} m_k, & k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}, \\ m_{w(k)}, & k \in \mathcal{S}^{\text{MTS}}_{\text{o}}. \end{cases} \tag{2}$$

Reallocation Activities. Reallocation activities involve fulfilling outstanding online MTS orders by transferring available inventory from the corresponding walk-in MTS buffer:[4] for $k \in \mathcal{S}^{\text{MTS}}_{\text{o}}$, activity k reallocates inventory from the walk-in buffer $w(k)$ to satisfy outstanding orders for product k. We assume reallocation activities are instantaneous. Let $\mu_k > 0$ denote the number of products reallocated per unit of reallocation activity k. That is, one good is reallocated from its walk-in MTS buffer to online MTS buffer k for every $1/\mu_k$ units of reallocation activity k undertaken. (This modeling convention facilitates the derivation of the approximating Brownian system but is otherwise inconsequential.) Let $S_k(t) \coloneqq \lfloor \mu_k t \rfloor$ denote the number of products reallocated as a result of the first t units of reallocation activity k undertaken.[5]

Dynamic scheduling decisions involve determining when and how much to engage in production and reallocation activities. We allow preemptive-resume scheduling and focus on head-of-line scheduling policies. A dynamic scheduling policy takes the form of a K-dimensional allocation process $T = \{T(t) : t \geq 0\}$, where each component $T_k(t)$ tracks the cumulative "effort" devoted to activity k up to time t. Specifically, for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$, $T_k(t)$ denotes the cumulative amount of time devoted to producing product k up to time

[4]In the context of a restaurant, this is akin to the manager allocating a completed item to an online order.

[5]Note that the units of production and reallocation activities differ in the sense that the argument of S_k for $k \in \mathcal{S}^{\text{MTS}}_{\text{o}}$ is the cumulative units of reallocation activity k undertaken, whereas for $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$ it is the total time the system manager has engaged in production activity k.

t, while for $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$, $T_k(t)$ denotes the cumulative units of reallocation activity k undertaken up to time t. Under a scheduling policy T, the cumulative server idleness up to time t is defined as follows:

$$I(t) := t - \sum_{k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} T_k(t), \quad t \geq 0. \tag{3}$$

We refer to $I = \{I(t) : t \geq 0\}$ as the cumulative idleness process associated with scheduling policy T. Note that production activities require time and consume server capacity, whereas reallocation activities are instantaneous and do not use server capacity; see Ata et al. (2020) for a similar treatment.

The system manager may reject customer orders to avoid incurring high abandonment and waiting costs when the system workload is deemed excessive. A dynamic rejection policy takes the form of a K-dimensional rejection process $R = \{R(t) : t \geq 0\}$, where $R_k(t)$ denotes the cumulative number of rejected orders for product $k \in \mathcal{S}$ up to time t. Let $r_k > 0$ denote the cost of rejecting a class k order.

3.3 State Dynamics

The system manager's dynamic control policy is a triple (λ, R, T), where λ is the instantaneous demand rate process, R is the rejection process, and T is the allocation process. We assume that demand for product $k \in \mathcal{S}$ arrives according to a nonhomogeneous Poisson process with instantaneous rate $\lambda_k(t)$; see Section 3.1. To be more specific, the cumulative demand for product k up to time t is given by $N_k\big(\int_0^t \lambda_k(s)\,ds\big)$, where N_k is a unit-rate Poisson process. Given the dynamic pricing and rejection decisions, we define the K-dimensional process $A = \{A(t) : t \geq 0\}$, which tracks the cumulative number of accepted orders up to time t, as follows: For $k \in \mathcal{S}$,

$$A_k(t) := N_k\Big(\int_0^t \lambda_k(s)\,ds\Big) - R_k(t), \quad t \geq 0, \tag{4}$$

where the first term on the right-hand side is the cumulative number of product k orders received up to time t, and the second term is the cumulative number of product k orders rejected up to time t.

We assume walk-in customers may abandon the system when faced with excessive waiting times, whereas online customers are assumed not to abandon.[6] Specifically, we assume that class $k \in \mathcal{S}_{\mathrm{w}}$ customers leave the system after an exponentially distributed amount of time with rate $\ell_k \geq 0$. Thus, the number of class k customers abandoned up to time t is given by $M_k(\int_0^t \ell_k Q_k^+(s)\,ds)$, where M_k is a unit-rate Poisson process.[7] We assume that the processes S_k, N_k, and M_k are mutually independent across product classes.

The system state is described by the K-dimensional queue length process $Q = \{Q(t), t \geq 0\}$, where $Q_k(t)$ denotes the number of outstanding (or backlogged) orders for product $k \in \mathcal{S}$ at time t. For $k \in \mathcal{S}^{\mathrm{MTO}}$, $Q_k(t)$ must remain nonnegative, as the production system cannot produce MTO products prior to receiving

[6]This reflects common practice where online customers prepay and cannot cancel after placing an order.
[7]The positive and negative parts of $x \in \mathbb{R}$ are denoted by $x^+ := \max\{x, 0\}$ and $x^- := \{-x, 0\}$, respectively.

an order. However, for $k \in \mathcal{S}^{\text{MTS}}$, $Q_k(t)$ can take negative values, reflecting a positive inventory of finished goods for the product. Without loss of generality, we assume the system manager reallocates goods to online MTS buffers only when they intend to immediately satisfy an order; hence, $Q_k(t)$ must remain nonnegative for all $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$ and $t \geq 0$.

Assuming the system is initially empty, the evolution of the queue length process Q under a control policy (λ, R, T) is characterized as follows: For MTO products $k \in \mathcal{S}^{\text{MTO}}$ and $t \geq 0$,

$$Q_k(t) \;=\; N_k\Big(\int_0^t \lambda_k(s)\,ds\Big) \;-\; S_k\big(T_k(t)\big) \;-\; M_k\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\text{w}}^{\text{MTO}}\}}\ell_k Q_k^+(s)\,ds\Big) \;-\; R_k(t), \quad t \geq 0, \tag{5}$$

where the first term on the right-hand side is the number of class k orders received up to time t, the second term is the number of class k products produced up to time t, the third term is the number of abandoned customers up to time t, and fourth term is the number of rejected customer orders up to time t. For walk-in MTS products $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$,

$$Q_k(t) \;=\; N_k\Big(\int_0^t \lambda_k(s)\,ds\Big) \;-\; S_k\big(T_k(t)\big) \;-\; M_k\Big(\int_0^t \ell_k Q_k^+(s)\,ds\Big) \;-\; R_k(t) \;+\sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)} S_j\big(T_j(t)\big), \quad t \geq 0, \tag{6}$$

where the first four terms on the right-hand side are analogous to those for MTO products. The fifth term, however, is the number of class k products reallocated to the online MTS buffers (sharing the same good with product k) up to time t. For online MTS products $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$,

$$Q_k(t) \;=\; N_k\Big(\int_0^t \lambda_k(s)\,ds\Big) \;-\; S_k\big(T_k(t)\big) \;-\; R_k(t), \quad t \geq 0, \tag{7}$$

where the first and third terms are analogous to those described above. The second term, however, is the number of products reallocated to online MTS class k from its corresponding walk-in buffer up to time t.

A dynamic control policy (λ, R, T) is said to be admissible if it is nonanticipating and satisfies

$$I \text{ is nondecreasing and continuous with } I(0) = 0, \tag{8}$$

$$A,\ R \text{ and } T \text{ are nondecreasing with } A(0) = R(0) = T(0) = 0, \tag{9}$$

$$\lambda(t) \in \mathcal{L} \text{ for all } t \geq 0, \tag{10}$$

$$\int_0^\infty \mathbb{1}_{\{Q_k(t)\geq 0\}}\,dT_j(t) = 0 \text{ for all } k \in \mathcal{S}_{\text{w}}^{\text{MTS}} \text{ and } j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k), \tag{11}$$

$$Q_k(t) \geq 0 \text{ for all } k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}} \text{ and } t \geq 0. \tag{12}$$

Equations (8)–(9) and (12) describe the inherent physical limitations governing the dynamics of A, I, R, T, and Q.[8] Equation (10) ensures that the instantaneous demand rate is achievable. Equation (11) ensures that goods are only reallocated when there is a positive inventory of finished goods at the walk-in buffer.

[8]The assumption that A is nondecreasing implies that customer orders can be rejected only upon their arrival. Our analysis does not rely on this assumption. Nonetheless, we adopt it since it is common in practice, e.g., restaurants decide whether to accept or reject customers upon their arrival.

3.4 Stochastic Control Problem

Tardiness costs are incurred when a product is not ready upon the customer's arrival; earliness costs are incurred when a product is produced before the customer arrives. To capture these costs in a unified manner, we define the cost function $\upsilon_k : \mathbb{R} \to \mathbb{R}_+$ for $k \in \mathcal{S}$ as follows:

$$\upsilon_k(x) := \begin{cases} \alpha_k x, & x \geq 0, \\ -\beta_k x, & x < 0, \end{cases} \tag{13}$$

where $\alpha_k > 0$ and $\beta_k \geq 0$ represent the per-unit tardiness and earliness costs for product k, respectively; see, e.g., Markowitz and Wein (2001) and Gao and Huang (2023). Tardiness costs apply to all products, while earliness costs apply only to online MTO products, as finishing their production prior to the customer's arrival is undesirable (e.g., due to loss of freshness). MTS products, by design, are intended for preproduction and thus are not subject to earliness costs. Accordingly, we set $\beta_k = 0$ for $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} \cup \mathcal{S}^{\mathrm{MTS}}$, and $\beta_k > 0$ otherwise. Let $w_k(t)$ denote the (virtual) sojourn time for a product k order arriving at time t. An earliness cost is incurred if $w_k(t) < \delta_k$, i.e., when the order is completed too early; a tardiness cost is incurred if $w_k(t) \geq \delta_k$, i.e., when the order is completed too late. It follows from (13) that the combined earliness and tardiness cost for a product k order arriving at time t is given by $\upsilon_k(w_k(t) - \delta_k)$.

Holding costs are incurred when there is an inventory of finished goods in the walk-in MTS buffers. Let $h_k > 0$ denote the per-unit holding cost for class $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ products. Abandonment costs are incurred when walk-in customers abandon. Let $d_k > 0$ denote the abandonment cost for class $k \in \mathcal{S}_{\mathrm{w}}$ customers. Rejection costs are incurred when orders are rejected. Let $r_k > 0$ denote the rejection cost of a class $k \in \mathcal{S}$ order.

Given an admissible control policy (λ, T, R), the cumulative profit up to time t is given as follows:

$$\begin{aligned} V(t) := \int_0^t \Pi\big(\lambda(s)\big)\, ds &- \sum_{k \in \mathcal{S}} \int_0^t \upsilon_k\big(w_k(s) - \delta_k\big)\, dA_k(s) - \sum_{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k Q_k^-(s)\, ds \\ &- \sum_{k \in \mathcal{S}_{\mathrm{w}}} d_k M_k\Big(\int_0^t \ell_k Q_k^+(s)\, ds \Big) - \sum_{k \in \mathcal{S}} r_k R_k(t), \quad t \geq 0. \end{aligned} \tag{14}$$

We refer to $V = \{V(t) : t \geq 0\}$ as the cumulative profit process. The first term on the right-hand side of (14) is a surrogate for the revenue (minus variable costs); see, e.g., Rubino and Ata (2009) and Ata and Barjesteh (2022). The second through fifth terms capture, respectively, the earliness and tardiness costs, holding costs, abandonment costs, and rejection costs. We next make the following natural assumption that ensures it is less costly for the system manager to reject a customer upfront than to risk customer dissatisfaction (due to a long waiting time) and an eventual abandonment.

Assumption 2. *For each $k \in \mathcal{S}_{\mathrm{w}}$, the rejection cost r_k is less than or equal to the abandonment cost d_k.*

Adopting the long-run average cost criterion, the system manager seeks to find a policy (λ, T, R) so as to

$$\text{maximize} \liminf_{t\to\infty} \frac{1}{t}\ \mathbb{E}\big[V(t)\big] \text{ subject to (3)–(12).} \tag{15}$$

Unfortunately, this formulation is a nonlinear, multi-dimensional stochastic control problem. Furthermore, when production times are not exponentially distributed, it is non-Markovian. As a result, the stochastic control problem is analytically intractable and suffers from the curse of dimensionality unless the total number of products is small. Therefore, we next consider a sequence of closely related systems in the (conventional) heavy-traffic regime and formulate an approximating Brownian control problem that is analytically tractable.

4 Approximating Brownian Control Problem

This section introduces a Brownian approximation for the control problem discussed in Section 3. Following Harrison (1988, 2000, 2003), we provide only enough detail to make plausible the proposed Brownian approximation and direct interested readers to these seminal papers for a comprehensive overview of the approximation technique we employ. In Brownian approximations, the original control problem is approximated by a related diffusion control problem, commonly referred to as the Brownian control problem, which emerges as the limit of a sequence of scaled systems indexed by the parameter n under conditions of heavy traffic. The original control problem corresponds to one such system, with an appropriately chosen system parameter n. The key assumption underlying the Brownian approximation is that the production and instantaneous demand rates in the original system are sufficiently large to enable us to approximate the various scaled processes of the original system by their counterparts in the Brownian approximation. Ultimately, this procedure allows us to interpret the solution of the Brownian control problem in the context of the original control problem.

4.1 Heavy Traffic Assumption and Asymptotic Regime

To facilitate the analysis to follow, we consider the following optimization problem, referred to as the static planning problem, which ignores randomness in the system.

Static Planning Problem. Choose an instantaneous demand rate vector λ so as to

$$\text{maximize } \Pi(\lambda) \text{ subject to } \lambda \in \mathcal{L}. \tag{16}$$

This formulation seeks to maximize the profit rate subject to the constraint $\lambda \in \mathcal{L}$, which ensures that the instantaneous demand rate vector λ is achievable. If the system manager were to disregard the randomness in the system—thereby ignoring all congestion-related costs—they would use an instantaneous demand rate that solves the static planning problem (16). However, due to randomness, they may benefit from dynamic adjustments to the instantaneous demand rate, which are captured by the Brownian approximation.

To develop the Brownian approximation, we consider a sequence of closely related systems indexed by a system parameter $n \in \mathbb{N}$. We attach a superscript n to various quantities of interest corresponding to the nth system in this sequence. We focus on the asymptotic regime where both demand and system capacity grow with n. Specifically, for $x \in \mathcal{P}$ and $n \in \mathbb{N}$,

$$\Lambda^n(x) \coloneqq n\Lambda(x) \quad \text{and} \quad \mu^n \coloneqq n\mu + \sqrt{n}\eta, \tag{17}$$

where $\mu, \eta \in \mathbb{R}_+^K$ are given constants. It follows directly from (17) that

$$(\Lambda^n)^{-1}(nx) = \Lambda^{-1}(x) \quad \text{and} \quad \Pi^n(nx) = n\Pi(x) \quad \text{for} \quad x \in \mathcal{L}. \tag{18}$$

Assumption 3 (Heavy Traffic Assumption)**.** *The static planning problem (16) has a unique optimal solution $\lambda^\star$. Moreover, $\lambda^\star \in \text{interior}(\mathcal{L})$ and satisfies $\sum_{k\in\mathcal{S}^{\text{MTO}}\cup\mathcal{S}_{\text{w}}^{\text{MTS}}} \rho_k = 1$, where ρ is the server utilization vector defined as follows:*

$$\rho_k \coloneqq \begin{cases} \lambda_k^\star/\mu_k, & k \in \mathcal{S}^{\text{MTO}}, \\ \left(\lambda_k^\star + \sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)} \lambda_j^\star\right)/\mu_k, & k \in \mathcal{S}_{\text{w}}^{\text{MTS}}. \end{cases} \tag{19}$$

Assumption 3 states that the profit-maximizing demand rate $\lambda^\star$ puts the system in heavy traffic. We refer to $\lambda^\star$ as the nominal instantaneous demand rate. (Since $\mathcal{L} \subset \mathbb{R}_+^K$, it follows from Assumption 3 that $\lambda_k^\star > 0$ for all $k \in \mathcal{S}$.) For $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, we interpret ρ_k as the proportion of time that, in the absence of randomness, should be allocated to producing class k products to meet the profit-maximizing demand. Recall that the server does not produce online MTS products directly. Instead, it reallocates finished MTS goods from the walk-in buffer to fulfill online orders. Consequently, for $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$, the term ρ_k accounts for the combined demand from both the walk-in and online channels. To ensure analytical tractability, we impose the following regularity condition on the profit rate function:

Assumption 4. *The profit rate function Π is twice continuously differentiable and strictly concave in a neighborhood of $\lambda^\star$.*

It follows from Assumption 3 and (18) that if the system manager were to ignore randomness in the system, they would choose the instantaneous demand rate $n\lambda^\star$. However, in the presence of randomness, it may be beneficial to dynamically adjust the instantaneous demand rate around $n\lambda^\star$. It is well known that in large balanced-flow systems in heavy traffic, the queue length process is of second-order (i.e., of order $\sqrt{n}$) relative to the system size (which is of order n). This suggests that fluctuations in the demand rate should also be of order $\sqrt{n}$ to effectively manage variability in the system. As such, we focus on instantaneous demand rate vectors of the following form: for $n \in \mathbb{N}$ and some $\zeta : [0, \infty) \to \mathbb{R}^K$,

$$\lambda^n(t) = n\lambda^\star + \sqrt{n}\,\zeta(t), \quad t \geq 0; \tag{20}$$

see Çelik and Maglaras (2008) and Ata and Barjesteh (2022) for similar treatments.

As discussed above, we expect both the queue length and rejection processes to be of second order relative to the system size in the heavy traffic regime. We therefore define the scaled queue length process $Z^n = \{Z^n(t) : t \geq 0\}$ and the scaled rejection process $O^n = \{O^n(t) : t \geq 0\}$ as follows:

$$Z^n(t) := \frac{Q^n(t)}{\sqrt{n}} \quad \text{and} \quad O^n(t) := \frac{R^n(t)}{\sqrt{n}}, \quad t \geq 0. \tag{21}$$

Furthermore, as thoroughly argued in Harrison (1988, 2000, 2003), any dynamic allocation policy worthy of consideration should satisfy the following when n is large:

$$T_k^n(t) \approx \begin{cases} \rho_k t, & k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}, \quad t \geq 0 \\ \lambda_k^\star t/\mu_k, & k \in \mathcal{S}_{\text{o}}^{\text{MTS}}, \qquad\qquad t \geq 0. \end{cases} \tag{22}$$

That is, for both MTO and walk-in MTS products, ρ_k should provide a first-order approximation to the fraction of time allocated to producing class k products. For online MTS products, $\lambda_k^\star/\mu_k$ should provide a first-order approximation to the amount of reallocation activity (or effort) required per unit of time to meet class k demand. However, due to randomness, it may be beneficial to make second-order deviations from these nominal values. To capture these deviations, we define the centered and scaled allocation process $Y^n = \{Y^n(t) : t \geq 0\}$ as follows: for $t \geq 0$,

$$Y_k^n(t) := \begin{cases} \sqrt{n}\big(\rho_k t - T_k^n(t)\big), & k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}, \\ \sqrt{n}\big(\lambda_k^\star t/\mu_k - T_k^n(t)\big), & k \in \mathcal{S}_{\text{o}}^{\text{MTS}}. \end{cases} \tag{23}$$

Finally, we define scaled idleness process $L^n = \{L^n(t) : t \geq 0\}$ as follows:

$$L^n(t) := \sqrt{n} I^n(t) = \sum_{k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}} Y_k^n(t), \quad t \geq 0. \tag{24}$$

4.2 Brownian Control Problem

We assume that the tardiness costs α_k^n, earliness costs β_k^n, holding costs h_k^n, abandonment costs d_k^n, and rejection costs r_k^n (for the product classes to which they apply) scale with n as follows:

$$\alpha_k^n := \frac{\alpha_k}{\sqrt{n}}, \quad \beta_k^n := \frac{\beta_k}{\sqrt{n}}, \quad h_k^n := \frac{h_k}{\sqrt{n}}, \quad d_k^n := \frac{d_k}{\sqrt{n}}, \quad \text{and} \quad r_k^n := \frac{r_k}{\sqrt{n}}, \tag{25}$$

where α_k, β_k, h_k, d_k, and r_k are nonnegative constants; see Harrison (2003), Çelik and Maglaras (2008), Ata and Barjesteh (2022), and Gao et al. (2023a) for similar treatments. Motivated by (13) and (25), we define the scaled cost function $\upsilon_k^n : \mathbb{R} \to \mathbb{R}_+$ for $k \in \mathcal{S}$ as follows:

$$\upsilon_k^n(x) := \frac{\upsilon_k(x)}{\sqrt{n}} = \begin{cases} \alpha_k^n x, & x \geq 0, \\ -\beta_k^n x, & x < 0. \end{cases} \tag{26}$$

As discussed above, in the heavy traffic asymptotic regime, the queue length process is of order $\sqrt{n}$, while the system capacity is of order n. Consequently, the sojourn times are expected to be of order $1/\sqrt{n}$. Therefore, we assume the quote times δ_k^n scale with n as follows:

$$\delta_k^n := \frac{\delta_k}{\sqrt{n}}, \quad k \in \mathcal{S}, \tag{27}$$

where δ_k is positive for online products and zero for walk-in products; see Rubino and Ata (2009) and Gao et al. (2023a) for a similar treatment. Finally, we assume the abandonment rates ℓ_k^n scale with n as follows:

$$\ell_k^n = \ell_k, \quad k \in \mathcal{S}_{\text{w}}, \tag{28}$$

where ℓ_k are nonnegative constants; see Rubino and Ata (2009) for a similar treatment.

If the system manager were to ignore the stochastic variability in the system—and thus choose the instantaneous demand rate $n\lambda^\star$ for the nth system—the cumulative profit up to time t would be $n\Pi(\lambda^\star)t$. The following result shows that the cumulative profit obtained in this deterministic system serves as an upper bound for the cumulative profit in the stochastic system under any admissible policy.

Proposition 1. *Under any admissible policy (λ^n, R^n, T^n), we have $V^n(t) \leq n\Pi(\lambda^\star)t$ for all $t \geq 0$ and sufficiently large n.*

Motivated by Proposition 1, we define the cumulative cost process $\xi^n = \{\xi^n(t) : t \geq 0\}$ as the deviation of the cumulative profit process V^n from that of the corresponding deterministic system. Specifically,

$$\xi^n(t) := n\Pi(\lambda^\star)t - V^n(t), \quad t \geq 0. \tag{29}$$

Following Harrison (1988, 2000, 2003), Appendix A formally derives the approximating Brownian control problem as n gets large. In the Brownian control problem, the processes Z_k^n, Y_k^n, I^n, R_k^n, and ξ^n are replaced by their formal limits Z_k, Y_k, L, O_k, and ξ, respectively, which jointly satisfy the following for $t \geq 0$:

$$\xi(t) = \int_0^t \zeta(s)' H \zeta(s)\, ds + \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(Z_k(s) - \lambda_k^\star \delta_k\big)\, ds + \sum_{k\in\mathcal{S}_{\text{w}}^{\text{MTS}}} \int_0^t h_k Z_k^-(s)\, ds$$
$$+ \sum_{k\in\mathcal{S}_{\text{w}}} \int_0^t d_k \ell_k Z_k^+(s)\, ds + \sum_{k\in\mathcal{S}} r_k O_k(t), \tag{30}$$

$$Z_k(t) = X_k(t) + \int_0^t \zeta_k(s)\, ds - \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\text{w}}^{\text{MTO}}\}} \ell_k Z_k^+(s)\, ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}^{\text{MTO}}, \tag{31}$$

$$Z_k(t) = X_k(t) + \int_0^t \zeta_k(s)\, ds - \int_0^t \ell_k Z_k^+(s)\, ds - O_k(t) + \mu_k Y_k(t) - \sum_{j\in\mathcal{S}_{\text{o}}^{\text{MTS}}(k)} \mu_j Y_j(t), \quad k \in \mathcal{S}_{\text{w}}^{\text{MTS}}, \tag{32}$$

$$Z_k(t) = X_k(t) + \int_0^t \zeta_k(s)\, ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}_{\text{o}}^{\text{MTS}}, \tag{33}$$

$$L(t) = \sum_{k\in\mathcal{S}^{\text{MTO}}\cup\mathcal{S}_{\text{w}}^{\text{MTS}}} Y_k(t), \tag{34}$$

L and O are nondecreasing with $L(0) = O(0) = 0$, (35)

$$Z_k(t) \geq 0 \text{ for all } k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}} \tag{36}$$

where $H \coloneqq -\nabla^2 \Pi(\lambda^\star)/2$, and $X_k = \{X_k(t) : t \geq 0\}$ for $k \in \mathcal{S}$ are independent Brownian motions with infinitesimal drift ν_k and infinitesimal variance σ_k^2 given by

$$\nu_k \coloneqq \begin{cases} -\eta_k \rho_k, & k \in \mathcal{S}^{\text{MTO}}, \\ -\eta_k \rho_k + \sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} \eta_j \lambda_j^\star/\mu_j, & k \in \mathcal{S}_{\text{w}}^{\text{MTS}}, \\ -\eta_k \lambda_k^\star/\mu_k, & k \in \mathcal{S}_{\text{o}}^{\text{MTS}}, \end{cases} \quad \text{and} \quad \sigma_k^2 \coloneqq \begin{cases} \lambda_k^\star \left(1 + c_{sk}^2\right), & k \in \mathcal{S}^{\text{MTO}}, \\ \lambda_k^\star + (\lambda_k^\star + \sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} \lambda_j^\star)\, c_{sk}^2), & k \in \mathcal{S}_{\text{w}}^{\text{MTS}}, \\ \lambda_k^\star, & k \in \mathcal{S}_{\text{o}}^{\text{MTS}}. \end{cases} \tag{37}$$

Equation (30), the counterpart to (14), states that cumulative cost process consists of five terms that capture the impact of pricing decisions, earliness and tardiness costs, holding costs, abandonment costs, and rejection costs, respectively. Equations (31), (32), and (33) are the counterparts to (5), (6), and (7). They describe the queue length dynamics by capturing the impact of pricing, abandonments, rejection, and scheduling decisions. Equation (34), the counterpart to (24), describes the server idleness in terms of the scheduling decisions. Finally, (35) and (36) are the natural counterparts to (8) and (12), respectively.

In the Brownian system, a control policy takes the form of a triple (ζ, O, Y), where ζ captures the dynamic adjustments to the pricing policy, O captures the dynamic order rejection policy, and Y captures the dynamic adjustments to the scheduling policy. A control policy (ζ, O, Y) is said to be admissible if it is nonanticipating, satisfies (30)–(36), and satisfies the following regularity condition:

$$\limsup_{t \to \infty} \frac{1}{t} \, \mathbb{E}\big[\|Z(t)\|\big] \; = \; 0, \tag{38}$$

where $\|\cdot\|$ denotes the Euclidean norm. The Brownian control problem (BCP) can now be stated as follows: Choose a nonanticipating control (ζ, Y, O) so as to

$$\text{minimize} \limsup_{t \to \infty} \frac{1}{t} \, \mathbb{E}\big[\xi(t)\big] \text{ subject to (30)–(38).} \tag{39}$$

Although the Brownian control problem is simpler than the original control problem due to the continuous nature of the state dynamics, it remains a multidimensional stochastic control problem. We next develop an equivalent one-dimensional formulation to the Brownian control problem and admits a closed-form solution.

5 Equivalent Workload Formulation

This section develops a one-dimensional stochastic control problem, referred to as the equivalent workload formulation, which is equivalent to the Brownian control problem for the purposes of optimal control. We define the one-dimensional workload process $W = \{W(t) : t \geq 0\}$ as follows:

$$W(t) \coloneqq \sum_{k \in \mathcal{S}} m_k Z_k(t), \quad t \geq 0. \tag{40}$$

The workload $W(t)$ can be interpreted as the total (scaled) backlog in the system at time t, which includes the work requested plus the work stored as finished goods inventory of the MTS products, measured in units of total work for the server. Substituting (31)–(33) into (40), note that

$$W(t) \;=\; B(t) + \int_0^t \theta(s)\, ds \;-\; \int_0^t \sum_{k \in \mathcal{S}_{\mathrm{w}}} m_k \ell_k Z_k^+(s)\, ds \;+\; L(t) \;-\; U(t), \quad t \geq 0, \tag{41}$$

where

$$B(t) \coloneqq \sum_{k\in\mathcal{S}} m_k X_k(t), \quad \theta(t) \coloneqq \sum_{k\in\mathcal{S}} m_k \zeta_k(t), \quad L(t) \coloneqq \sum_{k\in\mathcal{S}^{\mathrm{MTO}}\cup\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} Y_k(t), \quad U(t) \coloneqq \sum_{k\in\mathcal{S}} m_k O_k(t), \quad t \geq 0. \tag{42}$$

The process $B = \{B(t) : t \geq 0\}$ is a Brownian motion with infinitesimal drift $\mu \coloneqq \sum_{k\in\mathcal{S}} m_k \nu_k \leq 0$ and infinitesimal variance $\sigma^2 \coloneqq \sum_{k\in\mathcal{S}} m_k^2 \sigma_k^2 > 0$. The quantities $L(t)$ and $U(t)$ represent the cumulative (scaled) idleness and cumulative (scaled) work rejected up to time t, respectively. The processes $\theta = \{\theta(t) : t \geq 0\}$, $L = \{L(t) : t \geq 0\}$, and $U = \{U(t) : t \geq 0\}$ are referred to as the effective drift rate process, the effective idleness process, and the effective rejection process, respectively.

We next define the key components of the cost function in the workload formulation. First, we define the cost function $c : \mathbb{R} \to \mathbb{R}$ and the optimal drift rate function $\zeta^\star : \mathbb{R} \to \mathbb{R}^K$, respectively, as follows:

$$c(x) \coloneqq \min\left\{\zeta' H \zeta : m'\zeta = x,\, \zeta \in \mathbb{R}^K\right\} \quad \text{and} \quad \zeta^\star(x) \coloneqq \operatorname{argmin}\left\{\zeta' H \zeta : m'\zeta = x,\, \zeta \in \mathbb{R}^K\right\}, \quad x \in \mathbb{R}. \tag{43}$$

These functions specify the optimal cost and corresponding optimal drift rate vector for a given effective drift rate x. The following result provides closed-form expressions for these functions.

Lemma 1 (Ata and Barjesteh (2022, Lemma 1))**.** *The cost function $c : \mathbb{R} \to \mathbb{R}$ and the optimal drift rate function $\zeta^\star : \mathbb{R} \to \mathbb{R}^K$ can be equivalently expressed as follows:*

$$c(x) \;=\; \frac{x^2}{m' H^{-1} m} \quad \textit{and} \quad \zeta^\star(x) \;=\; \frac{H^{-1} m}{m' H^{-1} m} x, \quad x \in \mathbb{R}.$$

Next, we define the product class with the lowest rejection cost per unit of work and the effective rejection cost as follows:

$$k^\star \;\coloneqq\; \operatorname{argmin}\{r_k / m_k : k \in \mathcal{S}\} \quad \text{and} \quad \kappa \;\coloneqq\; r_{k^\star} / m_{k^\star}. \tag{44}$$

The effective rejection cost κ is the rejection cost associated with the greedy rejection policy that exclusively rejects orders for product $k^\star$, i.e., the product class with the most cost-effective rejection.

In the equivalent workload formulation, a control policy takes the form of a tuple (L, U, z, θ), where L captures the impact of idleness, U captures the impact of order rejections, $z : [0, \infty) \times \mathbb{R} \to \mathbb{R}^K$ captures the impact of scheduling (by specifying how to distribute the workload across the various product classes at each workload level over time), and θ captures the impact of pricing. We refer to the mapping z as the workload configuration function. Motivated by (36) and (40), we define

$$\mathcal{A}(w) \coloneqq \big\{z \in \mathbb{R}^K : m'z = w \text{ and } z_k \geq 0 \text{ for all } k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}\big\}, \quad w \in \mathbb{R}, \tag{45}$$

to be the set of admissible workload distribution vectors at workload level w. A control policy (L, U, z, θ) is said to be admissible to the equivalent workload formulation if it is nonanticipating and satisfies the following for $t \geq 0$:

$$\Xi(t) \;=\; \int_0^t c\big(\theta(s)\big)\,ds \;+\; \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(z_k(s, W(s)) - \lambda_k^\star \delta_k\big)\,ds \;+ \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k z_k^-(s, W(s))\,ds + \sum_{k\in\mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k z_k^+(s, W(s))\,ds \;+\; \kappa U(t), \tag{46}$$

$$W(t) \;=\; B(t) \;+\; \int_0^t \theta(s)\,ds \;-\; \int_0^t \sum_{k\in\mathcal{S}_{\mathrm{w}}} m_k \ell_k z_k^+(s, W(s))\,ds \;+\; L(t) \;-\; U(t), \tag{47}$$

$$z(t, W(t)) \in \mathcal{A}\big(W(t)\big) \text{ for all } t \geq 0, \tag{48}$$

$$L \text{ and } U \text{ are nondecreasing with } L(0) = U(0) = 0, \tag{49}$$

$$\limsup_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big[\|z(t, W(t))\|\big] \;=\; 0, \tag{50}$$

where B is a Brownian motion with infinitesimal drift μ and infinitesimal variance σ^2. The equivalent workload formulation (EWF) is as follows: Choose a nonanticipating control (L, U, z, θ) so as to

$$\text{minimize } \limsup_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big[\Xi(t)\big] \text{ subject to (46)–(50).} \tag{51}$$

The following result establishes the equivalence of the BCP (39) and the EWF (51):

Proposition 2. *For every admissible policy (ζ, O, Y) for the BCP (39), there exists an admissible policy (L, U, z, θ) for the EWF (51), and its cost is less than or equal to that of the policy (ζ, O, Y) for the BCP (39). Conversely, for every admissible policy (L, U, z, θ) for the EWF (51), there exists an admissible policy (ζ, O, Y) for the BCP (39), and and these two policies have the same cost.*

6 Solution to the Equivalent Workload Formulation

This section solves the workload formulation. To minimize technical complexity, we restrict attention to stationary Markov control policies, under which the workload configuration $z(t, W(t))$ and the effective drift rate $\theta(t)$ at time t depend on the history only through the current system workload $W(t)$. To reflect this assumption, we write $z(W(t))$ and $\theta(W(t))$ in place of $z(t, W(t))$ and $\theta(t)$, respectively. The remainder of this section is organized as follows. Section 6.1 introduces a class of barrier policies that play a central role in our analysis. Section 6.2 introduces the Bellman equation associated with the workload formulation. Finally, Section 6.3 solves the Bellman equation and uses its solution to derive an optimal solution to the workload formulation, which is a barrier policy.

6.1 Barrier Policies

Following Harrison (2013, Section 7.7), we define a barrier policy as follows:

Definition 1. Given policy parameters $l, u \in \mathbb{R}$ such that $l < u$, we call (L, U, z, θ) a barrier policy (with lower barrier at l and upper barrier at u) if it is an admissible control for the EWF (51) and satisfies $W(t) \in [l, u]$ for all $t \geq 0$, along with the following two conditions:

$$\int_0^t \mathbb{1}_{\{W(s)>l\}}\, dL(s) \;=\; 0 \quad \text{and} \quad \int_0^t \mathbb{1}_{\{W(s)<u\}}\, dU(s) \;=\; 0 \quad \text{for all} \quad t \geq 0.$$

Note that the processes L and U reflect the workload process at the lower barrier l and upper barrier u, respectively. For $n \in \mathbb{N}$, let $C^n[l, u]$ denote the space of functions $f : [l, u] \to \mathbb{R}$ that are n-times continuously differentiable; that is, all derivatives up to order n exist and are continuous on the closed interval $[l, u]$, including the boundary points. We next establish a method for computing the long-run average expected cost associated with a barrier policy. As a preliminarily to this result, given functions $z : \mathbb{R} \to \mathbb{R}^K$ and $\theta : \mathbb{R} \to \mathbb{R}$ and constant $\gamma \in \mathbb{R}$, define the second-order differential operator $\Gamma_{z,\theta}$ as follows:

$$\Gamma_{z,\theta} f(w) \;\coloneqq\; \frac{1}{2}\sigma^2 f''(w) \;+\; \big(\mu \;+\; \theta(w) \;-\sum_{k\in\mathcal{S}_{\mathrm{w}}} m_k \ell_k z_k^+(w)\big)\, f'(w), \quad (f, w) \in C^2[l, u] \times [l, u]. \tag{52}$$

We then consider the following second-order differential equation:

$$\Gamma_{z,\theta} f(w) \;+\; c(\theta(w)) \;+ \sum_{k\in\mathcal{S}} \upsilon_k\big(z_k(w) - \lambda_k^\star \delta_k\big) \;+ \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} h_k z_k^-(w) \;+ \sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k \ell_k z_k^+(w) \;=\; \gamma, \quad w \in [l, u], \tag{53}$$

subject to the boundary conditions

$$f'(l) \;=\; 0 \quad \text{and} \quad f'(u) \;=\; \kappa. \tag{54}$$

Proposition 3. *Consider an admissible barrier policy (L, U, z, θ) with a lower barrier at l and an upper barrier at u. If there exists $\gamma \in \mathbb{R}$ and $f \in C^2[l, u]$ that jointly satisfy (53)–(54), then γ is the long-run average expected cost associated with the barrier policy, i.e.,*

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\, \mathbb{E}\Big[\int_0^t c\big(\theta(W(s))\big)\, ds \;+ \sum_{k\in\mathcal{S}} \int_0^t \upsilon_k\big(z_k(W(s)) - \lambda_k^\star \delta_k\big)\, ds \;+ \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k z_k^-(W(s))\, ds \\ + \sum_{k\in\mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k z_k^+(W(s))\, ds \;+\; \kappa U(t)\Big] \;=\; \gamma.\end{aligned}$$

6.2 Bellman Equation

Motivated by Proposition 3 and smooth pasting arguments (see, e.g., Harrison (2013, Section 7.7)), this section introduces the Bellman equation corresponding to the workload formulation. The Bellman equation is introduced primarily to motivate our solution approach, and its properties that we require will be proved from first principles. As a preliminary, we define the following cost function, which captures the various state (i.e., earliness, tardiness, holding, and abandonment) costs:

$$\varphi(z,y) \coloneqq \sum_{k\in\mathcal{S}_{\mathrm{w}}} \ell_k\big(d_k - m_k y\big) z_k^+ + \sum_{k\in\mathcal{S}} \upsilon_k\big(z_k - \lambda_k^\star\delta_k\big) + \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} h_k z_k^-, \quad (z,y)\in\mathbb{R}^K\times\mathbb{R}. \tag{55}$$

The Bellman equation for the workload formulation is then given as follows: Find constants $l,u,\gamma\in\mathbb{R}$ and a function $f\in C^2[l,u]$ satisfying $l<u$ and

$$\min_{z\in\mathcal{A}(w),\, x\in\mathbb{R}} \left\{ \frac{1}{2}\sigma^2 f''(w) + (\mu + x) f'(w) + c(x) + \varphi\big(z, f'(w)\big) \right\} = \gamma, \quad w\in[l,u], \tag{56}$$

subject to the boundary conditions

$$f'(l) = f''(l) = f''(u) = 0 \quad \text{and} \quad f'(u) = \kappa. \tag{57}$$

The second-order boundary conditions in (57), i.e., $f''(l) = f''(u) = 0$, are known as smooth pasting conditions. They ensure the barriers l and u are chosen optimally. Motivated by Proposition 3, we interpret γ as a guess at the the minimum average cost and interpret l and u as the lower and upper reflecting barriers, respectively, for the workload process. The function f is often referred to as the relative value function in average cost dynamic programming.

The joint minimization over x and z in (56) can be decoupled. By Assumption 4 and Lemma 1, the minimization with respect to x is convex. Consequently, there exists a unique minimizer, given by

$$\operatorname*{argmin}_{x\in\mathbb{R}} \big\{ x f'(w) + c(x) \big\} = -\frac{m' H^{-1} m}{2} f'(w), \quad w\in[l,u]; \tag{58}$$

see, e.g., Ata and Barjesteh (2022, Lemma 2). To address the minimization over z, we define the effective state cost function $\phi:\mathbb{R}^2\to\mathbb{R}$ as follows:

$$\phi(w,y) \coloneqq \min_{z\in\mathcal{A}(w)} \varphi(z,y), \quad w,y\in\mathbb{R}. \tag{59}$$

Since (56)–(57) does not involve f itself, it is equivalent to a first-order differential equation. Letting $v = f'$, and using (58)–(59), we rewrite (56)–(57) as follows: Find constants $l,u,\gamma\in\mathbb{R}$ and a function $v\in C^1[l,u]$ satisfying $l<u$ and

$$\frac{1}{2}\sigma^2 v'(w) + \mu\, v(w) - \frac{m' H^{-1} m}{4} v^2(w) + \phi(w, v(w)) = \gamma, \quad w\in[l,u], \tag{60}$$

subject to the boundary conditions

$$v(l) \,=\, v'(l) \,=\, v'(u) \,=\, 0 \quad \text{and} \quad v(u) \,=\, \kappa. \tag{61}$$

6.3 Solution to the Bellman Equation

This section establishes the existence and uniqueness of a solution to the Bellman equation. Using this solution, we then propose a candidate barrier policy and prove that it is an optimal solution to the workload formulation. An explicit characterization of the candidate policy is provided in Appendix B.2.

The following result establishes the existence of a solution to the Bellman equation; its proof relies on several technical lemmas, which are deferred to Appendix B. As a preliminary, let

$$w_0 \,\coloneqq \sum_{k\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}} m_k\lambda_k^\star\delta_k \quad \text{and} \quad w_1 \,\coloneqq \sum_{k\in\mathcal{S}_{\mathrm{o}}} m_k\lambda_k^\star\delta_k, \tag{62}$$

and observe that $0 \le w_0 \le w_1$.[9] We interpret w_0 (resp., w_1) as the total workload in the online MTO classes (resp., all the online classes) assuming the head-of-line order in each class has a sojourn time equal to its quote time.

Theorem 1. *The Bellman equation (60)–(61) has a unique solution $(l^\star, u^\star, \gamma^\star, v^\star)$. Furthermore, $\gamma^\star > 0$, $l^\star < w_0 \le w_1 < u^\star$, and $v^\star \in C^1[l^\star, u^\star]$ is nonnegative and strictly increasing.*

In Appendix B.2, we prove Theorem 1 and provide a closed-form solution to the Bellman equation (60)–(61). To recover a solution to the original Bellman equation (56)–(57), we let

$$f^\star(w) \,\coloneqq \int_{l^\star}^{w} v^\star(x)\,dx, \quad w \in [l^\star, u^\star]. \tag{63}$$

Corollary 1. *The tuple $(l^\star, u^\star, \gamma^\star, f^\star)$ is the unique solution (up to an additive constant in $f^\star$) to the Bellman equation (56)–(57). Moreover, $f^\star$ is strictly increasing, strictly convex, and twice continuously differentiable.*

Equipped with a solution to the Bellman equation, we propose a candidate barrier policy for the EWF (51). In accordance with (58), we define the candidate effective drift rate function as follows:

$$\theta^\star(w) \,\coloneqq\, -\frac{m'H^{-1}m}{2}\, v^\star(w), \quad w \in [l^\star, u^\star]. \tag{64}$$

The candidate workload configuration is a minimizer of the right-hand side of (59) when $y = v^\star(w)$, and is not unique in general. In Appendix B.2.1, we provide a closed-form expression for our candidate workload

[9] We define $w_0 \coloneqq 0$ when $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} = \varnothing$ and $w_1 \coloneqq 0$ when $\mathcal{S}_{\mathrm{o}} = \varnothing$.

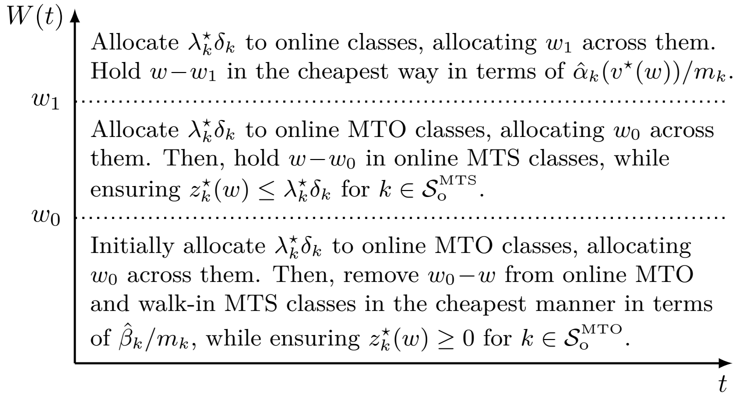

Figure 2: An illustrative description of the candidate workload configuration $z^\star(w)$.

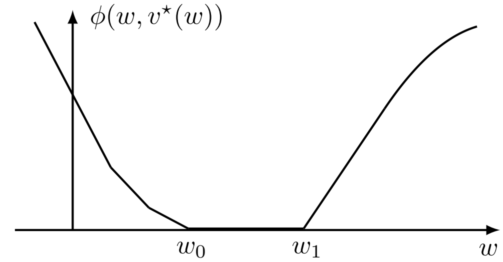

Figure 3: The effective state cost under the candidate workload configuration $\phi(w, v^\star(w))$.

configuration function. To be more specific, our candidate workload configuration function is given by

$$z^\star(w) := \mathcal{Z}(w, v^\star(w)) \in \underset{z \in \mathcal{A}(w)}{\operatorname{argmin}} \; \varphi(z, v^\star(w)), \quad w \in [l^\star, u^\star], \tag{65}$$

where $\mathcal{Z} : \mathbb{R}^2 \to \mathbb{R}$ is given by (110)–(112) in Appendix B.2.1. For the sake of completeness, we provide an informal description of $z^\star$ here. To facilitate this description, we rewrite the effective state cost function φ, defined in (55), in a more convenient form. Specifically, analogous to (13), we define the cost terms $\hat{\alpha}_k : \mathbb{R} \to \mathbb{R}$ and $\hat{\beta}_k \in \mathbb{R}$ for $k \in \mathcal{S}$ as follows:

$$\hat{\alpha}_k(y) := \begin{cases} \alpha_k + \ell_k(d_k - m_k y), & k \in \mathcal{S}_{\mathrm{w}}, \\ \alpha_k, & k \in \mathcal{S}_{\mathrm{o}}, \end{cases} \quad \text{and} \quad \hat{\beta}_k := \begin{cases} \beta_k, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}, \\ h_k, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \end{cases} \tag{66}$$

where $\hat{\alpha}_k(y)$ combines the tardiness and abandonment costs, and $\hat{\beta}_k$ combines the earliness and holding costs. We then define the corresponding cost functions $\hat{v}_k : \mathbb{R}^2 \to \mathbb{R}$ for $k \in \mathcal{S}$ (comprising tardiness, earliness, holding, and abandonment costs) as follows:

$$\hat{v}_k(x, y) := \begin{cases} \hat{\alpha}_k(y)\, x, & x \ge 0, \\ -\hat{\beta}_k\, x, & x < 0. \end{cases} \tag{67}$$

It then follows from (55) and (66)–(67) that φ can be rewritten as follows:

$$\varphi(z, y) = \sum_{k \in \mathcal{S}} \hat{v}_k(z_k - \lambda_k^\star \delta_k, y) = \sum_{k \in \mathcal{S}_{\mathrm{o}}} \hat{v}_k(z_k - \lambda_k^\star \delta_k, y) + \sum_{k \in \mathcal{S}_{\mathrm{w}}} \hat{v}_k(z_k, y), \quad (z, y) \in \mathbb{R}^K \times \mathbb{R}. \tag{68}$$

In this reformulation, all state costs associated with each product class k are consolidated into the single term $\hat{v}_k(z_k - \lambda_k^\star \delta_k, y)$. Moreover, it follows from (65) and (68) that the optimal workload configuration is the minimizer of the sum of these terms when $y = v^\star(w)$ at each workload level $w \in [l^\star, u^\star]$.

Figure 2 illustrates the structure of the candidate workload configuration, and Figure 3 shows the resulting effective state cost $\phi(w, v^\star(w))$. Notably, the structure of the workload configuration differs across the intervals $(-\infty, w_0)$, $[w_0, w_1]$, and (w_1, ∞), which we now describe in turn. For $w \in [w_0, w_1]$, each class

$k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}$ holds a workload of $m_k \lambda_k^\star \delta_k$, thereby holding w_0 across the online MTO classes. The remaining workload, $w - w_0$, is held in online MTS classes, with each class $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$ holding no more than $m_k \lambda_k^\star \delta_k$. No workload is held in walk-in classes. This workload configuration yields an effective state cost of zero.

For $w \in (-\infty, w_0)$, the workload is insufficient to achieve an effective state cost of zero. Initially, a workload of w_0 is allocated to the online MTO classes. This workload allocation has an effective state cost of zero. The (excess) workload of $w_0 - w$ is then deducted from the online MTO and walk-in MTS classes in the cheapest manner in terms of $\hat{\beta}_k / m_k$, which captures the earliness or holding cost per unit of work for class $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ products. This is done while ensuring that the workload in each of the online MTO classes remains nonnegative. No workload is held in the online MTS and walk-in MTO classes.

For $w \in (w_1, \infty)$, the workload is too large to achieve an effective state cost of zero. Initially, a workload of $m_k \lambda_k^\star \delta_k$ is allocated to each online class $k \in \mathcal{S}_{\mathrm{o}}$, thereby initially allocating a workload of w_1 across these classes. This workload allocation has an effective state cost of zero. The remaining workload of $w - w_1$ is then allocated to the cheapest class in terms of $\hat{\alpha}_k(v^\star(w))/m_k$, which reflects the tardiness and abandonment costs per unit of work for class $k \in \mathcal{S}$ products.

The following result provides an optimal solution to the workload formulation.

Theorem 2. *The barrier policy $(L^\star, U^\star, z^\star, \theta^\star)$ with lower barrier at $l^\star$, upper barrier at $u^\star$, effective drift rate function $\theta^\star$ defined by (64), and workload configuration function $z^\star$ defined by (65) is an optimal solution to the EWF (51), and it has a long-run average expected cost of $\gamma^\star$.*

The following corollary establishes the existence of an optimal solution to the BCP (39). An explicit characterization of this solution can be obtained by combining Theorem 2 with the solution provided in the proof of Proposition 2 in Appendix C.

Corollary 2. *There exists an optimal policy $(\zeta^\star, O^\star, Y^\star)$ to the BCP (39), and it has a long-run average expected cost of $\gamma^\star$.*

7 Proposed Policy

This section proposes a dynamic control policy for the problem introduced in Section 3 by interpreting the solution to the workload formulation in the context of the original control problem. The proposed policy consists of three components: pricing, rejection, and scheduling. Recall that we consider a sequence of systems indexed by n, whose formal limit is the BCP (39). The original control problem corresponds to a particular element in this sequence, determined by a specific choice of n. Consequently, to define the proposed policy, we first fix the system parameter n, which is used to unscale the processes of interest.

The proposed policy depends on the total system workload. We therefore define the unscaled (or nominal) workload process $W^n = \{W^n(t) : t \geq 0\}$ as $W^n(t) := \sum_{k \in \mathcal{S}} m_k Q_k^n(t)$ for $t \geq 0$, where $Q_k^n(t)$ denotes the queue length of product class k at time t; see Section 3. While the workload process in the equivalent workload formulation lives in $[l^\star, u^\star]$, the scaled nominal workload process $W^n(t)/\sqrt{n}$ in the original control problem may leave this interval. We therefore define $\mathcal{W}^n(t) := l^\star \vee (W^n(t)/\sqrt{n} \wedge u^\star)$ for $t \geq 0$, which projects $W^n(t)/\sqrt{n}$ into $[l^\star, u^\star]$, and express the proposed policy in terms of this projected workload process $\mathcal{W}^n(t)$.

Pricing Policy. To derive the proposed pricing policy, note that it follows from (42), (64), Lemma 1, and Corollary 2 that $\zeta^\star(w) = -(H^{-1}m)\, v^\star(w)$ for $w \in [l^\star, u^\star]$. Therefore, following (20), we choose the demand rate vector to be

$$\lambda^n(t) \;=\; n\lambda^\star \;-\; \sqrt{n}\,(H^{-1}m)\, v^\star(\mathcal{W}^n(t)), \quad t \geq 0. \tag{69}$$

Following (18), (69), Assumption 4, and Taylor's theorem, we then choose the price vector to be

$$p^n(t) \;=\; \Lambda^{-1}(\lambda^\star) \;-\; \frac{\nabla\Lambda^{-1}(\lambda^\star)}{\sqrt{n}}\,\frac{H^{-1}m}{2}\, v^\star(\mathcal{W}^n(t)), \quad t \geq 0, \tag{70}$$

where $\nabla\Lambda^{-1}(\lambda^\star)$ denotes the Jacobian of Λ^{-1} evaluated at $\lambda^\star$. Recall that the optimal effective drift rate control is nonnegative and strictly increasing in the workload. Thus, our proposed pricing policy induces an effective demand rate, $\sum_{k\in\mathcal{S}} \lambda^n_k(t)\, m_k$, that is strictly decreasing in the workload.

Rejection Policy. Recall that the optimal solution to the EWF (51) imposes an upper barrier on the workload process at $u^\star$ via the effective rejection process. In the context of the original control problem, we interpret this as the greedy rejection policy in which orders of product $k^\star$ are rejected when the nominal workload $W^n(t)$ is greater than or equal to the rejection threshold $u^n := \sqrt{n}u^\star$. (Here, $k^\star \in \mathcal{S}$ denotes the product class with the lowest rejection cost per unit of work; see (44).) The intuition behind this policy is that when the system backlog grows to the point where $W^n > u^n$, it becomes more cost-effective to reject additional orders rather than to allow the backlog to increase further.

Scheduling Policy. The proposed scheduling policy consists of two components: production and reallocation decisions. At each point in time, the policy first applies the reallocation decisions, followed by the production decisions. This policy is designed to distribute the workload in a manner that aligns with the optimal solution to the Brownian control problem. As is customary in the heavy traffic literature, we introduce small safety stocks to bridge the gap between the diffusion-scaled queue lengths prescribed by the optimal workload configuration function and the dynamics of the original system; see, e.g., Harrison (1996) and Maglaras (2000). The safety stock for product k is denoted by s_k for $k \in \mathcal{S}$. The safety stock values s_k can be calibrated via simulation; see, e.g., Plambeck (2004) and Ata and Barjesteh (2022).

Reallocation Decisions. Consider the product class $k \in \mathcal{S}^{\text{MTS}}_{\text{w}}$. If $Q^n_k(t) > 0$, no finished goods inventory is available for reallocation to the corresponding online classes. The proposed policy maintains a safety stock in product class k by choosing not to reallocate products when $\min(s_k, 0) \leq Q^n_k(t) \leq 0$. When $Q^n_k(t) < \min(s_k, 0)$, the proposed policy reallocates finished goods inventory from product class k to product classes $j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)$ for which $Q_j^n(t) > \sqrt{n} z_j^\star(\mathcal{W}^n(t)) + s_j$. Among such eligible online MTS product classes, it first prioritizes those for which $Q_j^n(t) > \sqrt{n}\lambda_j^\star \delta_k + s_j$, and then the remaining product classes. Within each group, products are ranked in descending order of their effective state costs per unit of work, i.e., $\hat{\alpha}_j\big(v^\star(\mathcal{W}^n(t))\big)/m_j$.

Production Decisions. The optimal solution to the workload formulation imposes a lower barrier on the workload process at $l^\star$ through the effective idleness process and distributes the workload according to the optimal workload configuration $z^\star$. Interpreting this in the context of the original control problem, the proposed policy idles the server if and only if the nominal workload is below the idling threshold $l^n := \sqrt{n}\, l^\star$ and the queue length is below the value prescribed by the optimal workload configuration plus the safety stock, i.e.,

$$W^n(t) \le \sqrt{n}\, l^\star \quad \text{and} \quad Q_k^n(t) \le \sqrt{n} z^\star(\mathcal{W}^n(t)) + s_k \quad \text{for all } k \in \mathcal{S}.$$

When the above condition does not hold, the server works. First, it aims to align the workload distribution with the optimal workload configuration. It does so by choosing the product class $k \in \mathcal{S}$ that has the largest change in the effective state cost per unit of work, which is given by

$$\tilde{v}_k(t) := \begin{cases} \hat{\alpha}_k(\mathcal{W}^n(t))/m_k, & \text{if } z_k^\star(\mathcal{W}^n(t)) \ge \lambda_k^\star \delta_k, \\ -\hat{\beta}_k/m_k, & \text{otherwise,} \end{cases}$$

among the products for which the queue length is above the value prescribed by the optimal workload configuration plus the safety stock, i.e., those satisfying $Q_k^n(t) > \sqrt{n} z^\star(\mathcal{W}^n(t)) + s_k$. If such a product exits, the server produces the corresponding good (for online MTS products, once the good is produced, it is added to the corresponding walk-in buffer).

If, however, the workload distribution is aligned with the optimal workload configuration, i.e., $Q_k^n(t) \le z_k^\star(\mathcal{W}^n(t)) + s_k$ for all $k \in \mathcal{S}$ and $W^n(t) > l^n$, the server focuses solely on the product class that results in the largest decrease (equivalently, smallest increase) in the effective state cost per unit of work. This product class is given by $k^\star(\mathcal{W}^n(t), v^\star(\mathcal{W}^n(t)))$; for the precise definition of this product class, see (107) in Appendix B.2. Specifically, when $w \in (w_1, \infty)$, it produces the product class that results in the largest decrease in the effective state cost; when $w \in (-\infty, w_0)$, it produces the online MTO or walk-in MTS product class that results in the smallest increase in the effective state cost; and when $w \in [w_0, w_1]$, it produces the good associated with an online MTS product class that has a positive queue length.

8 Simulation Study

This section illustrates the effectiveness of our proposed policy through a simulation study and provides several managerial insights, with implications for omnichannel restaurants. We consider an omnichannel production system with $K = 4$ products, consisting of one MTS good and one MTO good, each available through both walk-in and online channels. The online channel operates with a single 30-minute quote time. Therefore, we have $\mathcal{S} = \{1, 2, 3, 4\}$. For the purposes of the simulation study, we number the products such that $\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} = \{1\}$, $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} = \{2\}$, $\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}} = \{3\}$, and $\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}} = \{4\}$. Activities are numbered in a manner consistent with the product numbering: for $k = 1, 2, 3$, activity k corresponds to the production of product k, while activity 4 corresponds to the reallocation of finished MTS goods from the walk-in buffer to fulfill online orders. The remainder of the section is organized as follows. Section 8.1 describes the demand model used in the simulation study. Section 8.2 outlines the parameters of the base example. Section 8.3 presents the results and managerial insights. Additional graphs and tables are provided in Appendix D.3.

8.1 Nested Logit Demand Model

We model customer demand using a nested logit model (Gallego and Topaloglu, 2019, Section 8.6.3). In a nested logit model, a customer first selects a nest of products and then chooses a specific product within that nest. In our setting, the nests correspond to the ordering channels—namely, online and walk-in. Let $\mathcal{N}_i$ denote the set of products in nest $i \in \{\text{Online}, \text{Walk-in}\}$. The probability that a customer selects nest i given the price vector p is denoted by $\mathbb{P}(\text{nest } i \,|\, p)$. Conditional on choosing nest i and the price vector p, the probability that a customer selects product $k \in \mathcal{N}_i$ is denoted by $\mathbb{P}(\text{product } k \,|\, \text{nest } i,\, p)$. Following Gallego and Topaloglu (2019), these probabilities are given as follows: For $i \in \{\text{Online}, \text{Walk-in}\}$ and $k \in \mathcal{N}_i$,

$$\mathbb{P}(\text{nest } i \,|\, p) \;=\; \frac{e^{c_i V_i}}{1 + \sum_{i \in \{\text{Online}, \text{Walk-in}\}} e^{c_i V_i}} \quad \text{and} \quad \mathbb{P}(\text{product } k \,|\, \text{nest } i,\, p) \;=\; \frac{e^{a_k - b_k p_k}}{\sum_{l \in \mathcal{N}_i} e^{a_l - b_l p_l}},$$

where $a_k \in \mathbb{R}$ captures the inherent "value" of product k, $b_k \geq 0$ denotes its price sensitivity, $V(i) \coloneqq \log(\sum_{l \in \mathcal{N}_i} e^{a_l - b_l p_l})$ represents the value (or attractiveness) of nest i, and $c_i \in (0, 1]$ captures the degree of substitution within nest i. By the law of total probability, the probability that a customer selects product $k \in \mathcal{N}_i$ given the price vector p is given by

$$\mathbb{P}(\text{product } k \,|\, p) \;=\; \mathbb{P}(\text{nest } i \,|\, p)\; \mathbb{P}(\text{product } k \,|\, \text{nest } i,\, p).$$

We assume that each unit of time corresponds to one hour in our simulation study. Moreover, customers arrive at the rate of $\Lambda_0 > 0$ and choose a product according to the probabilities given above. Then, the demand function is given as follows:

$$\Lambda_k(p) \;=\; \Lambda_0\, \mathbb{P}(\text{product } k \,|\, p), \quad k \in \mathcal{S}, \quad p \in \mathbb{R}_+^K.$$

8.2 System Parameters in the Base Case

We now specify the parameters of the base case. To assess the robustness of our findings and to extract structural insights, we later perform a sensitivity analysis across several key model parameters. We set

$\Lambda_0 = 666.67$, $a_k = -0.99$, $b_k = 1.79$, and $c_i = 1$ for $k \in \mathcal{S}$ and $i \in \{\text{Online}, \text{Walk-in}\}$. For simplicity, we assume all products have identical variable costs of production and set $\gamma_k = 0.3$ for $k \in \mathcal{S}$. These parameters yield a nominal demand rate vector of $\lambda^\star = (33.33, 33.33, 33.33, 33.33)$ and a nominal static price vector of $p^\star = (1, 1, 1, 1)$. Under the nominal prices, a customer purchases a product with probability 0.2 and, conditional on making a purchase, selects each product with equal probability.[10] Given these parameters, the total variable cost of production is 30% of the revenue, aligning with industry statistics (TouchBistro, 2025). In the restaurant industry, fixed costs typically account for approximately 60% of the revenue (TouchBistro, 2025), corresponding to a fixed cost of production of 80 per unit of time in our example. Thus, accounting for both fixed and variable costs of production, the profit rate in the absence of randomness is 13.33.

The production rates are set to $\mu_1 = \mu_2 = 100$ and $\mu_3 = 200$. Hence, the server spends approximately $\rho_k = 1/3$ fraction of its time producing each product $k \in \{1, 2, 3\}$. The abandonment rates are set to zero, i.e., $\ell_k = 0$ for $k \in \mathcal{S}_\text{w}$. The abandonment costs are set to $d_k = (p_k^\star - \gamma_k) + 0.1 = 0.8$ for $k \in \mathcal{S}_\text{w} = \{1, 3\}$. Similarly, thE rejection costs are set to $r_k = 0.8$ for $k \in \mathcal{S}$. The (per-unit) tardiness costs of all products are the same and the (per-unit) earliness cost of the online MTO product is set to half of its tardiness cost, i.e., $\alpha_k = 0.05$ for $k \in \mathcal{S}$ and $\beta_4 = 0.025$. The holding cost of the walk-in MTS product is set to a quarter of its tardiness costs, i.e., $h_3 = 0.25\alpha_3 = 0.0125$. Finally, the system (scaling) parameter is set to $n = 100$. Given these model parameters, it follows that $l^\star = -0.06$, $w_0 = 1.67$, $w_1 = 2.50$, $u^\star = 8.16$ and $k^\star = 1$.

8.3 Results and Managerial Insights

Ideally, the performance of the proposed policy should be compared against a benchmark to validate its effectiveness. However, there is no exact method for solving the original control problem due to the presence of the sojourn times in the second term of (14). Nevertheless, as is standard in the heavy-traffic literature (see, e.g., Plambeck et al. (2001), Ata (2006), and Rubino and Ata (2009)), one can obtain a reasonable approximation to the original problem by replacing $w_k(t)$ with $Q_k(t)/\lambda_k^\star$ for $k \in \mathcal{S}$ and $t \geq 0$. This approximation yields a semi-Markov decision process (SMDP) formulation of the joint dynamic pricing, scheduling, and rejection control problem; see Appendix D.1 for details. Therefore, we compare the performance of our proposed policy with the optimal solution of this surrogate control problem—henceforth referred to as the SMDP policy with dynamic prices. This policy can only be computed numerically when the number of products is small, as it suffers from the curse of dimensionality; see Appendix D.2 for details.

To further examine the performance of our proposed policy and quantify the value of dynamic pricing, we compare our proposed policy—also referred to as the BCP policy with dynamic prices—with three additional policies. The first policy—referred to as the SMDP policy with static prices—optimizes the rejection and scheduling decisions dynamically under the best static price vector. Specifically, given a static price vector,

[10] Several reports, including the one referenced in Korniichuk and Boryczka (2021), cite conversion rates in the restaurant industry to be in the range of 15% to 25% for online customers.

we solve the SMDP described in Appendix D.1 to find the optimal rejection and scheduling decisions. Then, we use gradient descent to find the static price vector that minimizes the long-run average cost of the SMDP.

The second policy—referred to as the BCP policy with static prices—uses a static price vector in conjunction with dynamic rejection and scheduling policies similar to our proposed policy. Specifically, it is the policy corresponding to the optimal solution to the workload formulation with a static drift rate. To construct this policy, we solve the Bellman equation corresponding to a workload formulation with a given static drift rate. Then, we use gradient descent to find the static drift rate that minimizes the long-run average cost of the workload formulation.

The third policy—referred to as the BCP policy with online-only dynamic prices—uses dynamic pricing, rejection, and scheduling policies similar to our proposed policy, except the prices of the walk-in products are static. This policy corresponds to the optimal solution to the Brownian control problem with a static drift rate for the walk-in products. To construct this policy, for a given drift rate vector for the walk-in products, we solve the resulting workload formulation. Then, we use gradient descent to find the static drift rate vector for the walk-in products that minimizes the long-run average cost of this workload formulation.

We simulate each policy 100 times for 10^6 time units starting with an initial queue length of zero for all products. To eliminate the transient effects, the initial 20% of each run is discarded. Table 1 in Appendix D.3 reports the mean and 95% confidence interval for the prices used by each policy in the base case. Across all policies and products, prices remain within 13% of the nominal static prices. This highlights that the price adjustments required for dynamic pricing are modest, aligning with regulatory preferences that discourage large price swings (Clyne-Canham, 2023a). The BCP policies with static and dynamic prices use prices that are close to those of their SMDP counterparts. The static pricing policies use prices that are modestly higher (about 1–2%) than the average prices used by the dynamic pricing policies. However, dynamic pricing policies exhibit greater dispersion: while the average price is lower, some customers face prices up to 10% higher. Finally, static pricing is operationally less attractive—customers pay higher prices on average, and as we will see later, businesses incur higher costs.

We conduct a sensitivity analysis across several key parameters. Figure 4 reports the long-run average cost of various policies across a range of parameter values. Tables 2-7 in Appendix D.3 report both the long-run average costs and the gaps relative to the SMDP policy with dynamic prices. Our proposed policy maintains a consistently small gap relative to the SMDP policy with dynamic prices (less than 5% in all instances), and it outperforms the static pricing policies by a significant margin in almost all instances. The considerable gap between the proposed policy and the SMDP policy with static prices (approximately 38% in the base case) highlights the value of dynamic pricing. The small gap between the BCP policy with static prices and the SMDP policy with static prices (less than 5% in almost all instances) highlights the small sub-optimality of our scheduling, rejection, and static pricing decisions. While our comparative analysis highlights the superior performance of the proposed policy, it also yields broader insights into when dynamic

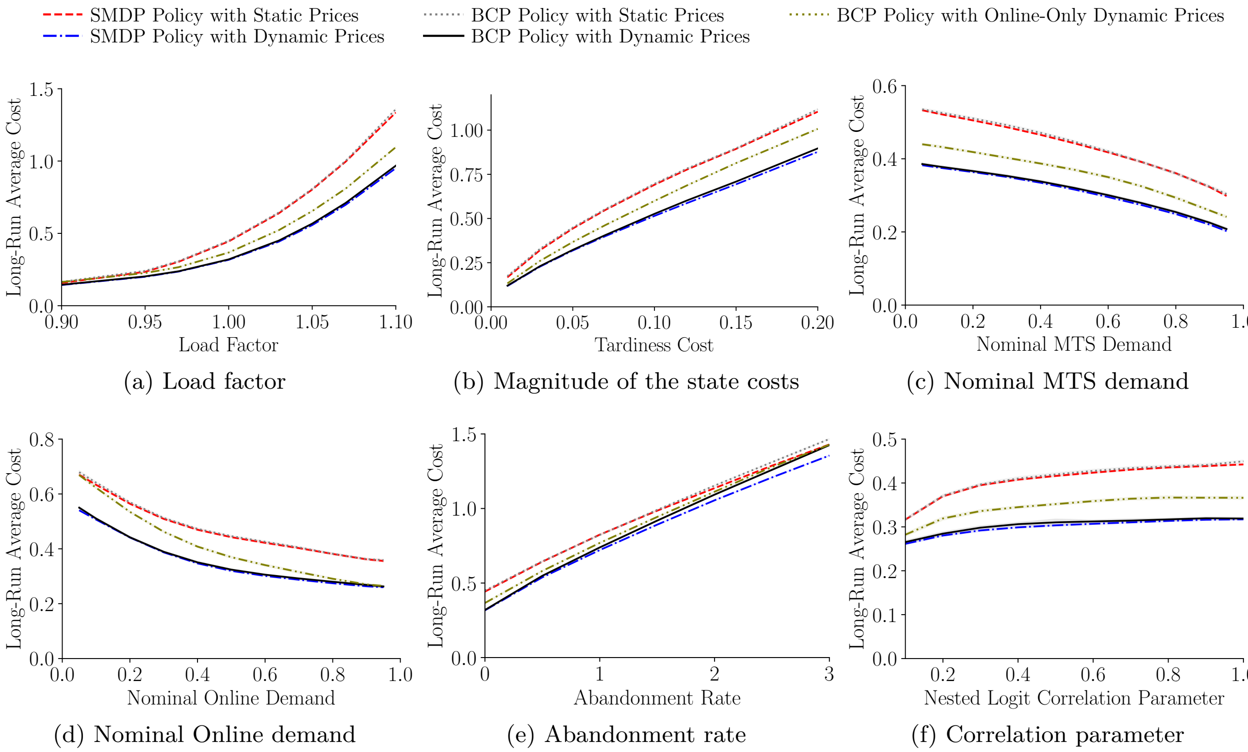

Figure 4: Impact of the load factor, magnitude of the state (earliness, tardiness, and holding) costs, fraction of the demand choosing the MTS good under $p^\star$ (i.e., nominal MTS demand), fraction of the demand choosing the online channel under $p^\star$ (i.e., nominal online demand), abandonment rate, and the nested logit correlation parameter on the performance of the various policies. The shaded areas depict the 95% confidence intervals.

pricing delivers greater value (relative to static pricing). By value, we refer to the relative (percentage) performance improvement of dynamic pricing policies.

First, we examine the impact of the load factor $\sum_{k\in\mathcal{S}} \lambda_k^\star/\mu_k$ by varying the rate at which customers arrive Λ_0. Figure 4a shows that as the load factor increases, the long-run average cost of all the policies increases. However, the value of dynamic pricing relative to static pricing increases; see Table 2 in Appendix D.3. This is because as the load factor increases, so does the (average) workload, thereby increasing the value of policies that can dynamically adjust the demand rate to mitigate congestion.

Second, we examine the impact of the state (earliness, tardiness, and holding) costs by varying the magnitude of the tardiness costs α_k proportionally while holding the ratios β_k/α_k and h_k/α_k fixed. As depicted in Figure 4b, increasing the state costs increases the long-run average cost of all policies. Our proposed policy maintains a small gap relative to the SMDP policy with dynamic prices across all tardiness cost values. Moreover, the value of dynamic pricing (relative to static pricing) remains consistently high (over 25%) across all tardiness cost values; see Table 3 in Appendix D.3.

Third, we examine the impact of demand composition—specifically, the fraction of customers choosing the MTS good and the fraction choosing the online channel. To that end, we vary the demand function parameters to change the nominal MTS demand, i.e., $\sum_{k\in\mathcal{S}_{\mathrm{w}}} \lambda_k^\star$, and the nominal online demand, i.e.,

$\sum_{k \in \mathcal{S}_o} \lambda_k^\star$, while keeping the total demand rate, i.e., $\sum_{k \in \mathcal{S}} \lambda_k^\star$, and the nominal static prices fixed, thereby keeping the nominal profit rate fixed. Figures 4c and 4d show that as either the nominal MTS or online demand increase, the long-run average cost of all the policies decreases. This can be attributed to the additional flexibility conferred by the MTS good and the online channel. Moreover, the value of dynamic pricing (relative to static pricing) increases with either the nominal MTS or online demand; see Tables 4 and 5 in Appendix D.3. This is because, as either quantity increases, the system spends less time in states with high tardiness costs, increasing the value of policies that can dynamically adjust the demand rate.

Our findings carry important managerial insights and implications regarding the impact of demand composition. First, businesses with substantial online demand—or the ability to shift demand online—stand to benefit most from dynamic pricing. A similar, albeit more modest, conclusion applies to businesses with high MTS demand. Second, as the nominal online demand increases, the gap between the fully dynamic pricing policies and the BCP policy with online-only dynamic prices decreases.[11] This suggests that dynamically pricing the online channel only can be a suitable entry point for adopting dynamic pricing, particularly for businesses with a strong online channel, such as many restaurants. This policy enables such business to realize significant gains by starting with the online channel while gradually developing the capabilities needed to dynamically price the walk-in channel.

Fourth, we examine the impact of customer abandonments. To that end, we vary the abandonment rates ℓ_1 and ℓ_3, while maintaining $\ell_1 = \ell_3$. Figure 4e shows that as the abandonment rates increase, the long-run average cost of all policies increases, while the value of dynamic pricing (relative to static pricing) decreases; see Table 6 in Appendix D.3. Abandonments reduce the effective load on the system, thereby decreasing the average workload and diminishing the value of policies that can dynamically adjust the demand rate. In other words, abandonments act as a partial substitute for dynamic pricing by reducing system congestion. Finally, we examine the impact of the correlation parameter in the nested logit model. Figure 4f shows that the value of dynamic pricing (relative to static pricing) decreases as the correlation parameter decreases; see Table 7 in Appendix D.3. That is, as products within a nest become closer substitutes, the value of dynamic pricing decreases.

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[11]When more than 70% of the nominal demand is online, this gap is less than 10%; see Figure 4d.

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A Formal Derivation of the Approximating Brownian System

This section presents a formal derivation for the approximating Brownian control problem discussed in Section 4. The analysis given below does not constitute a rigorous convergence proof of the pre-limit production system to its Brownian approximation. However, the arguments made in support of the Brownian approximation can be viewed as a broad outline of such a proof.

There are two common approaches used to formally derive approximating Brownian systems: weak convergence arguments and functional strong approximations. The weak convergence approach involves arguing that the distribution of a sequence of centered and scaled stochastic processes converges to the distribution of a limiting process; see, e.g., Harrison (1988, 2003). On the other hand, the functional strong approximation approach involves arguing the almost sure convergence of a sequence of stochastic processes to a limiting process; see, e.g., Çelik and Maglaras (2008) and Ata and Tongarlak (2013). Although both methods produce the same approximating Brownian system, as a deliberate effort to align with the seminal works of J. Michael Harrison, we opt for a weak convergence argument.

The general procedure is as follows: First, we consider a sequence of systems indexed by n, operating under conditions of heavy traffic. Second, we center and scale the processes of interest. Finally, we let n get large and replace these processes with their formal limits; see Harrison (1988, 2003) for a more detailed explanation of this procedure. Before proceeding with the analysis, we impose two technical assumptions that underlie the mathematical development. First, all random elements are defined on a fixed probability space. Second, all continuous-time stochastic processes have sample paths that are right-continuous with finite left limits (RCLL).

Fluid and Diffusion Scaled Processes. We begin by defining the fluid-scaled and diffusion-scaled processes. For $k \in \mathcal{S}$, the fluid-scaled instantaneous demand rate process is defined as

$$\bar{\lambda}_k^n(t) := \frac{\lambda_k^n(t)}{n} = \lambda_k^\star + \frac{\zeta_k(t)}{\sqrt{n}}, \quad t \geq 0. \tag{71}$$

For $k \in \mathcal{S}$, the fluid-scaled queue length process is defined as

$$\bar{Q}_k^n(t) := \frac{Q_k^n(t)}{n} = \frac{Z_k^n(t)}{\sqrt{n}}, \quad t \geq 0. \tag{72}$$

For $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, the diffusion-scaled production process is defined as

$$\hat{S}_k^n(t) := \frac{S_k^n(t) - \mu_k^n t}{\sqrt{n}}, \quad t \geq 0. \tag{73}$$

For $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$, the scaled (deterministic) reallocation process is defined as

$$\hat{S}_k^n(t) := \frac{\lfloor \mu_k^n t \rfloor - \mu_k^n t}{\sqrt{n}}, \quad t \geq 0. \tag{74}$$

For $k \in \mathcal{S}$, the diffusion-scaled Poisson process for order arrivals is defined as

$$\hat{N}_k^n(t) \coloneqq \frac{N_k(nt) - nt}{\sqrt{n}}, \quad t \geq 0. \tag{75}$$

For $k \in \mathcal{S}_{\mathrm{w}}$, the diffusion-scaled Poisson process for customer abandonments is defined as

$$\hat{M}_k^n(t) \coloneqq \frac{M_k(nt) - nt}{\sqrt{n}}, \quad t \geq 0. \tag{76}$$

Note from the development in Section 3.2 that $S_k = \{S_k(t) : t \geq 0\}$ is a renewal process with rate μ_k for each $k \in \mathcal{S}$. It follows from the functional central limit theorem for renewal processes and the independence of the processes S, N, and M that the diffusion-scaled processes $\hat{S}^n$, $\hat{N}^n$, and $\hat{M}^n$ converge in distribution to independent multi-dimensional Brownian motions (of appropriate dimension) with independent components; see Billingsley (1999, Section 14) for details.

Scaled Queue Length Process for the MTO Products. We next rewrite the scaled queue length processes for the MTO products: For $k \in \mathcal{S}^{\mathrm{MTO}}$ and $t \geq 0$, we have that

$$\begin{aligned}
Q_k^n(t) &= N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) - S_k^n\big(T_k^n(t)\big) - M_k\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k^n\big[Q_k^n(s)\big]^+ ds\Big) - R_k^n(t) \\
&= N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) + \Big[n\int_0^t \bar{\lambda}_k^n(s)\,ds - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] \\
&\quad - S_k^n\big(T_k^n(t)\big) + \Big[\mu_k^n T_k^n(t) - \mu_k^n T_k^n(t)\Big] - M_k\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big) \\
&\quad + \Big[\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds - \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big] - R_k^n(t) \\
&= \Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] + n\int_0^t \bar{\lambda}_k^n(s)\,ds - \Big[S_k^n\big(T_k^n(t)\big) - \mu_k^n T_k^n(t)\Big] - \mu_k^n T_k^n(t) \\
&\quad - \Big[M_k\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big) - \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big] \\
&\quad - \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds - R_k^n(t),
\end{aligned} \tag{77}$$

where the first equality follows from (5), the second equality from (28) and (71), and the third equality by the associative property. Dividing both sides of (77) by $\sqrt{n}$ and using (21) gives

$$\begin{aligned}
Z_k^n(t) &= \frac{1}{\sqrt{n}}\Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) - n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] + \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds - \frac{1}{\sqrt{n}}\Big[S_k^n\big(T_k^n(t)\big) - \mu_k^n T_k^n(t)\Big] \\
&\quad - \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) - \frac{1}{\sqrt{n}}\Big[M_k\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big) - \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds\Big] \\
&\quad - \frac{1}{\sqrt{n}}\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Q_k^n(s)\big]^+ ds - \frac{1}{\sqrt{n}}R_k^n(t) \\
&= \hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) + \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds - \hat{S}_k^n\big(T_k^n(t)\big) - \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) + \frac{1}{\sqrt{n}}\Big[\mu_k^n\rho_k t - \mu_k^n\rho_k t\Big] \\
&\quad - \hat{M}_k^n\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[\bar{Q}_k^n(s)\big]^+ ds\Big) - \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}}\ell_k\big[Z_k^n(s)\big]^+ ds - O_k^n(t),
\end{aligned} \tag{78}$$

where the second equality follows from (21), (72)–(73), and (75). Then, rearranging the terms on the right-hand side of (78) gives

$$
\begin{aligned}
Z_k^n(t) \,&=\, \Big[\hat N_k^n\Big(\int_0^t \bar\lambda_k^n(s)\,ds\Big) \,-\, \hat S_k^n\big(T_k^n(t)\big) \,-\, \hat M_k^n\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\rm w}^{\rm MTO}\}}\ell_k\big[\bar Q_k^n(s)\big]^+ds\Big)\Big]\\
&\qquad +\, \sqrt{n}\int_0^t \bar\lambda_k^n(s)\,ds \,-\, \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\rm w}^{\rm MTO}\}}\ell_k\big[Z_k^n(s)\big]^+ds \,-\, \frac{1}{\sqrt{n}}\mu_k^n\rho_k t\\
&\qquad -\, O_k^n(t) \,-\, \frac{1}{\sqrt{n}}\mu_k^n\Big[T_k^n(t) \,-\, \rho_k t\Big]\\
&=\, \Big[\hat N_k^n\Big(\int_0^t \bar\lambda_k^n(s)\,ds\Big) \,-\, \hat S_k^n\big(T_k^n(t)\big) \,-\, \hat M_k^n\Big(\int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\rm w}^{\rm MTO}\}}\ell_k\big[\bar Q_k^n(s)\big]^+ds\Big) \,-\, \eta_k\rho_k t\Big]\\
&\qquad +\, \sqrt{n}(\lambda_k^\star \,-\, \mu_k\rho_k)t \,+\, \int_0^t \zeta_k(s)\,ds \,-\, \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\rm w}^{\rm MTO}\}}\ell_k\big[Z_k^n(s)\big]^+ds\\
&\qquad -\, O_k^n(t) \,+\, \Big(\mu_k \,+\, \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t),
\end{aligned}
\tag{79}
$$

where the second equality follows from (23), (71), and rearranging terms. Finally, we define the process $\{X_k^n(t) : t \geq 0\}$ for $k \in \mathcal{S}^{\rm MTO}$ as follows:

$$
X_k^n(t) \,=\, \hat N_k^n\Big(\int_0^t \bar\lambda_k^n(s)\,ds\Big) \,-\, \hat S_k^n\big(T_k^n(t)\big) \,-\, \hat M_k^n\Big(\int_0^t \ell_k\mathbb{1}_{\{k\in\mathcal{S}_{\rm w}^{\rm MTO}\}}\big[\bar Q_k^n(s)\big]^+ds\Big) \,-\, \eta_k\rho_k t. \tag{80}
$$

Using the definition of ρ_k in (19) and substituting (80) into (79), we obtain the following expression for the scaled queue length process for the MTO products:

$$
Z_k^n(t) \,=\, X_k^n(t) + \int_0^t \zeta_k(s)\,ds - \int_0^t \mathbb{1}_{\{k\in\mathcal{S}_{\rm w}^{\rm MTO}\}}\ell_k\big[Z_k^n(s)\big]^+ds \,-\, O_k^n(t) \,+\, \Big(\mu_k \,+\, \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t). \tag{81}
$$

Scaled Queue Length Process for the Walk-In MTS Products. We next rewrite the scaled queue length processes for the walk-in MTS products: For $k \in \mathcal{S}_{\rm w}^{\rm MTS}$ and $t \geq 0$, we have that

$$
\begin{aligned}
Q_k^n(t) \,&=\, N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) \,-\, S_k^n\big(T_k^n(t)\big) \,-\, M_k\Big(\int_0^t \ell_k^n\big[Q_k^n(s)\big]^+ds\Big) \,-\, R_k^n(t) + \sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)} S_j^n\big(T_j^n(t)\big)\\
&=\, N_k\Big(n\int_0^t \bar\lambda_k^n(s)\,ds\Big) \,+\, \Big[n\int_0^t \bar\lambda_k^n(s)\,ds \,-\, n\int_0^t \bar\lambda_k^n(s)\,ds\Big] \,-\, S_k^n\big(T_k^n(t)\big) \,+\, \Big[\mu_k^n T_k^n(t) \,-\, \mu_k^n T_k^n(t)\Big]\\
&\qquad -\, M_k\Big(\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big) \,+\, \Big[\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds - \int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big] \,-\, R_k^n(t)\\
&\qquad + \sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)} S_j^n\big(T_j^n(t)\big) + \sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)} \Big[\mu_j^n T_j^n(t) \,-\, \mu_j^n T_j^n(t)\Big]\\
&=\, \Big[N_k\Big(n\int_0^t \bar\lambda_k^n(s)\,ds\Big) \,-\, n\int_0^t \bar\lambda_k^n(s)\,ds\Big] \,+\, n\int_0^t \bar\lambda_k^n(s)\,ds \,-\, \Big[S_k^n\big(T_k^n(t)\big) \,-\, \mu_k^n T_k^n(t)\Big] \,-\, \mu_k^n T_k^n(t)\\
&\qquad -\, \Big[M_k\Big(\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big) - \int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big] - \int_0^t \ell_k\big[Q_k^n(s)\big]^+ds \,-\, R_k^n(t),\\
&\qquad + \sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)} \Big[S_j^n\big(T_j^n(t)\big) \,-\, \mu_j^n T_j^n(t)\Big] + \sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)} \mu_j^n T_j^n(t),
\end{aligned}
\tag{82}
$$

where the first equality follows from (6), the second equality from (28) and (71), and the third equality by the associative property. Dividing both sides of (82) by $\sqrt{n}$ and using (21) gives

$$\begin{aligned}
Z_k^n(t) \;=\; &\frac{1}{\sqrt{n}}\Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \;-\; n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] \;+\; \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds \;-\; \frac{1}{\sqrt{n}}\Big[S_k^n\big(T_k^n(t)\big) \;-\; \mu_k^n T_k^n(t)\Big] \\
&-\; \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) \;-\; \frac{1}{\sqrt{n}}\Big[M_k\Big(\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big) \;-\; \int_0^t \ell_k\big[Q_k^n(s)\big]^+ds\Big] \;-\; \frac{1}{\sqrt{n}}\int_0^t \ell_k\big[Q_k^n(s)\big]^+ds \\
&-\; \frac{1}{\sqrt{n}}R_k^n(t) \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\frac{1}{\sqrt{n}}\Big[S_j^n\big(T_j^n(t)\big) \;-\; \mu_j^n T_j^n(t)\Big] \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\frac{1}{\sqrt{n}}\mu_j^n T_j^n(t) \\
=\; &\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \;+\; \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds \;-\; \hat{S}_k^n\big(T_k^n(t)\big) \;-\; \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) \;+\; \frac{1}{\sqrt{n}}\Big[\mu_k^n\rho_k t \;-\; \mu_k^n\rho_k t\Big] \\
&-\; \hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) \;-\int_0^t \ell_k\big[Z_k^n(s)\big]^+ds \;-\; O_k^n(t) \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\hat{S}_j^n\big(T_j^n(t)\big) \\
&+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\frac{1}{\sqrt{n}}\mu_j^n T_j^n(t) \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\frac{1}{\sqrt{n}}\Big[\mu_j^n\lambda_j^\star\mu_j^{-1}t \;-\; \mu_j^n\lambda_j^\star\mu_j^{-1}t\Big],
\end{aligned} \tag{83}$$

where the second equality follows from (21) and (72)–(75). Then, rearranging the terms on the right-hand side of (83) gives

$$\begin{aligned}
Z_k^n(t) \;=\; &\Big[\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \;-\; \hat{S}_k^n\big(T_k^n(t)\big) \;-\; \hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\hat{S}_j^n\big(T_j^n(t)\big)\Big] \\
&+\; \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds \;-\int_0^t \ell_k\big[Z_k^n(s)\big]^+ds \;-\; \frac{1}{\sqrt{n}}\mu_k^n\rho_k t \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\frac{1}{\sqrt{n}}\mu_j^n\lambda_j^\star\mu_j^{-1}t \;-\; O_k^n(t) \\
&-\; \frac{1}{\sqrt{n}}\mu_k^n\Big[T_k^n(t) \;-\; \rho_k t\Big] \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\frac{1}{\sqrt{n}}\mu_j^n\Big[T_j^n(t) \;-\; \lambda_j^\star\mu_j^{-1}t\Big] \\
=\; &\Bigg[\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \;-\; \hat{S}_k^n\big(T_k^n(t)\big) \;-\; \hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\hat{S}_j^n\big(T_j^n(t)\big) \\
&+\; \Big(-\,\eta_k\rho_k \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\eta_j\lambda_j^\star\mu_j^{-1}\Big)t\Bigg] \;+\; \sqrt{n}\Big(\lambda_k^\star \;+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\mu_j\lambda_j^\star\mu_j^{-1} \;-\; \mu_k\rho_k\Big)t \;+\int_0^t \zeta_k(s)\,ds \\
&-\int_0^t \ell_k\big[Z_k^n(s)\big]^+ds \;-\; O_k^n(t) \;+\; \Big(\mu_k \;+\; \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t) \;-\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\Big(\mu_j \;+\; \frac{1}{\sqrt{n}}\eta_j\Big)Y_j^n(t),
\end{aligned} \tag{84}$$

where the second equality follows from (23), (71), and rearranging terms. Finally, we define the process $X_k^n = \{X_k^n(t) : t \geq 0\}$ for $k \in \mathcal{S}_{\rm w}^{\rm MTS}$ as follows:

$$\begin{aligned}
X_k^n(t) \;=\; &\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \;-\; \hat{S}_k^n\big(T_k^n(t)\big) \;-\; \hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ds\Big) \\
&+\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\hat{S}_j^n\big(T_j^n(t)\big) \;-\; \Big(\eta_k\rho_k \;-\sum_{j\in\mathcal{S}_{\rm o}^{\rm MTS}(k)}\eta_j\lambda_j^\star\mu_j^{-1}\Big)t.
\end{aligned} \tag{85}$$

Using the definition of ρ_k in (19) and substituting (85) into (84), we obtain the following expression for the scaled queue length process for the walk-in MTS products:

$$
\begin{aligned}
Z_k^n(t) \,&=\, X_k^n(t) \,+\, \int_0^t \zeta_k(s)\,ds \,-\, \int_0^t \ell_k\big[Z_k^n(s)\big]^+ ds \,-\, O_k^n(t) \\
&\quad + \Big(\mu_k \,+\, \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t) - \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \Big(\mu_j \,+\, \frac{1}{\sqrt{n}}\eta_j\Big)Y_j^n(t).
\end{aligned} \tag{86}
$$

Scaled Queue Length Process for the Online MTS Products. We next rewrite the scaled queue length processes for the online MTS products: For $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$ and $t \geq 0$, we have that

$$
\begin{aligned}
Q_k(t) \,&=\, N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) \,-\, S_k^n\big(T_k^n(t)\big) \,-\, R_k^n(t) \\
&= N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \,+\, \Big[n\int_0^t \bar{\lambda}_k^n(s)\,ds \,-\, n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] \,-\, S_k^n\big(T_k^n(t)\big) \\
&\quad + \Big[\mu_k^n T_k^n(t) \,-\, \mu_k^n T_k^n(t)\Big] \,-\, R_k^n(t) \\
&= \Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \,-\, n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] \,+\, n\int_0^t \bar{\lambda}_k^n(s)\,ds \\
&\quad - \Big[S_k^n\big(T_k^n(t)\big) \,-\, \mu_k^n T_k^n(t)\Big] \,-\, \mu_k^n T_k^n(t) \,-\, R_k^n(t),
\end{aligned} \tag{87}
$$

where the first equality follows from (6), the second equality from (71), and the third equality by the associative property. Dividing both sides of (87) by $\sqrt{n}$ and using (21) gives

$$
\begin{aligned}
Z_k^n(t) \,&=\, \frac{1}{\sqrt{n}}\Big[N_k\Big(n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \,-\, n\int_0^t \bar{\lambda}_k^n(s)\,ds\Big] \,+\, \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds \\
&\quad - \frac{1}{\sqrt{n}}\Big[S_k^n\big(T_k^n(t)\big) \,-\, \mu_k^n T_k^n(t)\Big] \,-\, \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) \,-\, \frac{1}{\sqrt{n}}R_k^n(t) \\
&= \hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \,+\, \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds \,-\, \hat{S}_k^n\big(T_k^n(t)\big) \,-\, \frac{1}{\sqrt{n}}\mu_k^n T_k^n(t) \\
&\quad + \frac{1}{\sqrt{n}}\Big[\mu_k^n \lambda_k^\star \mu_k^{-1} t \,-\, \mu_k^n \lambda_k^\star \mu_k^{-1} t\Big] \,-\, O_k^n(t),
\end{aligned} \tag{88}
$$

where the second equality follows from (21) and (74)–(75). Then, rearranging the terms on the right-hand side of (88) gives

$$
\begin{aligned}
Z_k^n(t) \,&=\, \Big[\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \,-\, \hat{S}_k^n\big(T_k^n(t)\big)\Big] \,+\, \sqrt{n}\int_0^t \bar{\lambda}_k^n(s)\,ds \,-\, \frac{1}{\sqrt{n}}\mu_k^n \lambda_k^\star \mu_k^{-1} t \\
&\quad - O_k^n(t) \,-\, \frac{1}{\sqrt{n}}\mu_k^n\Big[T_k^n(t) \,-\, \lambda_k^\star \mu_k^{-1} t\Big] \\
&= \Big[\hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \,-\, \hat{S}_k^n\big(T_k^n(t)\big) \,-\, \eta_k \lambda_k^\star \mu_k^{-1} t\Big] \,+\, \sqrt{n}\big(\lambda_k^\star \,-\, \mu_k \lambda_k^\star \mu_k^{-1}\big)t \,+\, \int_0^t \zeta_k(s)\,ds \\
&\quad - O_k^n(t) \,+\, \Big(\mu_k \,+\, \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t),
\end{aligned} \tag{89}
$$

where the second equality follows from (23), (71), and rearranging terms. Finally, we define the process $\{X_k^n(t) : t \geq 0\}$ for $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$ as follows:

$$X_k^n(t) \,=\, \hat{N}_k^n\Big(\int_0^t \bar{\lambda}_k^n(s)\,ds\Big) \,-\, \hat{S}_k^n\big(T_k^n(t)\big) \,-\, \eta_k\lambda_k^\star\mu_k^{-1}t. \tag{90}$$

By canceling terms and substituting (90) into (89), we obtain the following expression for the scaled queue length process for the online MTS products:

$$Z_k^n(t) \,=\, X_k^n(t) \,+\int_0^t \zeta_k(s)\,ds \,-\, O_k^n(t) \,+\, \Big(\mu_k \,+\, \frac{1}{\sqrt{n}}\eta_k\Big)Y_k^n(t). \tag{91}$$

Cumulative Cost Process. We now derive an expression for the cumulative cost process. Recall that the cumulative profit process $V^n = \{V^n(t) : t \geq 0\}$ for the nth system is given by

$$\begin{aligned} V^n(t) \,=& \int_0^t \Pi^n\big(\lambda^n(s)\big)\,ds \,-\, \sum_{k\in\mathcal{S}}\int_0^t \upsilon_k^n\big(w_k^n(s)-\delta_k^n\big)\,dA_k^n(s) \,-\sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}}\int_0^t h_k^n\big[Q_k^n(s)\big]^-ds \\ &-\sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k^n M_k\Big(\int_0^t \ell_k^n\big[Q_k^n(s)\big]^+ds\Big) \,-\, \sum_{k\in\mathcal{S}} r_k^n R_k^n(t), \end{aligned} \tag{92}$$

where the process $A_k^n = \{A_k^n(t) : t \geq 0\}$ tracks the number of admitted class $k \in \mathcal{S}$ orders over time in the nth system, and it is given by

$$A_k^n(t) \,=\, N_k\Big(\int_0^t \lambda_k^n(s)\,ds\Big) \,-\, R_k^n(t). \tag{93}$$

We rewrite (92) by considering each term on the right-hand side individually. For the first term on the right-hand side of (92), applying (18) and (20) gives

$$\Pi^n\big(\lambda^n(s)\big) \,=\, \Pi^n\big(n\lambda^\star + \sqrt{n}\zeta(s)\big) \,=\, n\Pi\Big(\lambda^\star + \frac{1}{\sqrt{n}}\zeta(s)\Big). \tag{94}$$

Since $\lambda^\star \in \mathrm{interior}(\mathcal{L})$ and Π is twice continuously differentiable on $\mathcal{L}$ (see Assumptions 3 and 4), for sufficiently large n, the multivariate version of Taylor's theorem yields

$$\begin{aligned} \Pi\Big(\lambda^\star + \frac{1}{\sqrt{n}}\,\zeta(s)\Big) \,&=\, \Pi(\lambda^\star) \,+\, \frac{1}{\sqrt{n}}\nabla\Pi(\lambda^\star)'\zeta(s) \,+\, \frac{1}{2n}\,\zeta(s)'\,\nabla^2\Pi(\lambda^\star)\,\zeta(s) \,+\, o(1/n) \\ &=\, \Pi(\lambda^\star) \,+\, \frac{1}{2n}\,\zeta(s)'\,\nabla^2\Pi(\lambda^\star)\,\zeta(s) \,+\, o(1/n), \end{aligned} \tag{95}$$

where the second equality follows from $\nabla\Pi(\lambda^\star) = 0$ (see Assumption 3). Substituting (95) into (94) and integrating both sides then gives

$$\int_0^t \Pi^n\big(\lambda^n(s)\big)\,ds \,=\, n\Pi(\lambda^\star)t \,+\, \frac{1}{2}\int_0^t \zeta(s)'\,\nabla^2\Pi(\lambda^\star)\,\zeta(s)\,ds \,+\, o(1/n). \tag{96}$$

For the second term on the right-hand side of (92), letting $\bar{A}_k^n(t) := n^{-1}A_k^n(t)$ for $t \geq 0$ and applying (26) gives

$$\int_0^t \upsilon_k^n\big(w_k^n(s) - \delta_k^n\big)\, dA_k^n(s) \;=\; \frac{1}{\sqrt{n}}\int_0^t \upsilon_k\big(w_k^n(s) - \delta_k^n\big)\, dA_k^n(s) \;= \int_0^t \upsilon_k\big(\sqrt{n}w_k^n(s) - \delta_k\big)\, d\bar{A}_k^n(s),$$

where the second equality follows from (27). Finally, for the fourth term on the right-hand side of (92), applying (21), (25), (28), and (74) gives

$$\sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k^n M_k\Big(\int_0^t \ell_k^n\big[Q_k^n(s)\big]^+ ds\Big) \;=\; \sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k\Big[\hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ ds\Big) \;+\; \int_0^t \ell_k\big[Z_k^n(s)\big]^+ ds\Big]. \tag{97}$$

Then, by combining (25)–(29), (92), (96), and (97), we obtain the following expression for the cumulative cost process:

$$\begin{aligned}\xi^n(t) \;=\; &-\frac{1}{2}\int_0^t \zeta(s)'\nabla^2\Pi(\lambda^\star)\zeta(s)\, ds \;+\sum_{k\in\mathcal{S}}\int_0^t \upsilon_k\big(\sqrt{n}w_k^n(s) - \delta_k\big)\, d\bar{A}_k^n(s) \;+\sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}}\int_0^t h_k\big[Z_k^n(s)\big]^- ds \\ &+\sum_{k\in\mathcal{S}_{\mathrm{w}}}\int_0^t d_k\ell_k\big[Z_k^n(s)\big]^+ ds \;+\sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k\hat{M}_k^n\Big(\int_0^t \ell_k\big[\bar{Q}_k^n(s)\big]^+ ds\Big) \;+\; \sum_{k\in\mathcal{S}} r_k O_k^n(t) \;+\; o(1).\end{aligned} \tag{98}$$

Approximating Brownian Control Problem. We established above that the scaled queue length processes for the MTO, walk-in MTS, and online MTS products satisfy (80)–(81), (85)–(86), and (90)–(91), respectively, and that the cumulative cost process satisfies (98). We now provide a formal limiting argument to justify approximating these processes with their diffusion limits as n gets large. As articulated in Section 4.1 (in particular, see (22)), a key assertion in this approximation is that as n grows large, the only allocation policies worthy of consideration are those satisfying

$$T_k^n(t) \approx \rho_k t, \quad k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}} \qquad \text{and} \qquad T_k^n(t) \approx \lambda_k^\star \mu_j^{-1} t, \quad k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}.$$

We now argue that the K-dimensional process $X^n = (X_k^n)$, given by (80), (85), and (90), can be approximated by a K-dimensional Brownian motion X. By (71), observe that $\bar{\lambda}^n \approx \lambda^\star$ as n gets large. Applying the functional central limit theorem for renewal processes and (a loose application of) the random time change theorem, we approximate the first terms on the right-hand side of (80), (85), and (90) by $B_k(\lambda_k^\star t)$ for $k \in \mathcal{S}$, where B_k is a standard Brownian motion; see, e.g., Billingsley (1999). By (22), a similar reasoning applies to the second terms on the right-hand side of (80) and (85) to argue that as n gets large,

$$\hat{S}_k^n(T_k^n(t)) \;\approx\; \begin{cases} \widetilde{B}_k\big(\lambda_k^\star(1+\xi_k^2)t\big), & k \in \mathcal{S}^{\mathrm{MTO}}, \\ \widetilde{B}_k\big((\lambda_k^\star + \xi_k^2\mu_k\rho_k)t\big), & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \end{cases}$$

where $\widetilde{B}_k$ is a standard Brownian motion. Moreover, since the queue length processes are expected to be of order $\sqrt{n}$ in the heavy-traffic regime, it follows that $\bar{Q}_k^n(t) \approx 0$ for $k \in \mathcal{S}$ as n gets large; see (72). Finally, by (74), it is immediate that $\hat{S}_k^n(T_k^n(t))$ for $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$ converges to zero almost surely as n gets large. We conclude that the third term on the right-hand side of (80), the third and fourth terms on the right-hand side of (85), and the second term on the right-hand side of (90) vanish as n gets large. This argument formally justifies the approximation $X_k^n \approx X_k$ for $k \in \mathcal{S}$ as n gets large, where the X_k are independent (ν_k, σ_k^2) Brownian motions with infinitesimal drift and infinitesimal variance, respectively, given as follows:

$$\nu_k := \begin{cases} -\eta_k \rho_k, & k \in \mathcal{S}^{\mathrm{MTO}}, \\ -\eta_k \rho_k + \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \eta_j \lambda_j^\star / \mu_j, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ -\eta_k \lambda_k^\star / \mu_k, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \end{cases} \qquad \sigma_k^2 := \begin{cases} \lambda_k^\star \left(1 + c_{sk}^2\right), & k \in \mathcal{S}^{\mathrm{MTO}}, \\ \lambda_k^\star + (\lambda_k^\star + \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \lambda_j^\star)\, c_{sk}^2, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ \lambda_k^\star, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}. \end{cases}$$

Using the formal limiting arguments above, we replace the processes Z_k^n, X_k^n, Y_k^n, I^n, and O_k^n in the nth system with their formal limits Z_k, X_k, Y_k, L, and O_k, respectively, which jointly satisfy the following:

$$\begin{aligned} Z_k(t) &= X_k(t) + \int_0^t \zeta_k(s)\, ds - \int_0^t \mathbb{1}_{\{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}} \ell_k Z_k^+(s) ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}^{\mathrm{MTO}}, \\ Z_k(t) &= X_k(t) + \int_0^t \zeta_k(s)\, ds - \int_0^t \ell_k Z_k^+(s)\, ds - O_k(t) + \mu_k Y_k(t) - \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \mu_j Y_j(t), \quad k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \\ Z_k(t) &= X_k(t) + \int_0^t \zeta_k(s)\, ds - O_k(t) + \mu_k Y_k(t), \quad k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \\ L(t) &= \sum_{k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} Y_k(t) \end{aligned}$$

Similarly, (8)–(9) and (11)–(12) translate to the following conditions:

$$\begin{aligned} &L \text{ and } O \text{ are nondecreasing with } L(0) = O(0) = 0, \\ &Z_k(t) \geq 0 \text{ for all } k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}. \end{aligned}$$

Finally, we consider the cumulative cost process in (98). Observe that the second term on the right-hand side of (98) contains the sojourn time term $w_k^n(t)$ for a product $k \in \mathcal{S}$ order accepted at time t. Applying the "snapshot principle" of Reiman (1984), we approximate $w_k^n(t)$ as follows:

$$\sqrt{n} w_k^n(t) \approx \frac{Q_k^n(t)}{\sqrt{n} \lambda_k^\star} = \frac{Z_k^n(t)}{\lambda_k^\star},$$

which becomes more accurate as the system approaches heavy traffic. The snapshot principle states that in heavy traffic, the state of the system changes negligibly during the time an order spends in the system; see, e.g., Ata and Tongarlak (2013), Kim et al. (2018), and Liu and Sun (2022) for similar applications of the snapshot principle. Furthermore, it follows from the functional strong law of large numbers, (a loose application of) the random time change theorem, and the fact that the rejection process is expected to be of order $\sqrt{n}$ in the heavy-traffic regime that $\bar{A}_k^n(t) \approx \lambda_k^\star t$ as n gets large. Therefore, as n gets large,

$$\int_0^t \upsilon_k\big(\sqrt{n} w_k^n(s) - \delta_k\big)\, d\bar{A}_k^n(s) \approx \int_0^t \upsilon_k\Big(\frac{Z_k^n(s)}{\lambda_k^\star} - \delta_k\Big)\, d(\lambda_k^\star s) = \int_0^t \upsilon_k\big(Z_k^n(s) - \lambda_k^\star \delta_k\big)\, ds.$$

Using the above formal limiting arguments, we replace the cumulative cost process ξ^n in the nth with its limiting process ξ, which satisfies the following:

$$\xi(t) \;=\; -\frac{1}{2}\int_0^t \zeta(s)'\nabla^2\Pi(\lambda^\star)\zeta(s)\,ds \;+\; \sum_{k\in\mathcal{S}}\int_0^t \upsilon_k\big(Z_k(s)-\lambda_k^\star\delta_k\big)\,ds \;+\sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}}\int_0^t h_k Z_k^-(s)\,ds$$

$$+\sum_{k\in\mathcal{S}_{\mathrm{w}}}\int_0^t d_k\ell_k Z_k^+(s)\,ds \;+\; \sum_{k\in\mathcal{S}} r_k O_k(t).$$

B Solving the Bellman Equation

This section establishes the existence and uniqueness of the solution to the Bellman equation (56)–(57), effectively proving Theorem 1. We also provide an explicit characterization of the optimal value function and an optimal workload configuration function. To this end, we first analyze a class of Riccati equations in Section B.1, which includes the Bellman equation, and provide an explicit solution for this class of equations. We then apply these results in Section B.2 to construct a solution to the Bellman equation.

B.1 Solution to a Special Class of Riccati Equations

In this section, we analyze an initial value problem (IVP) associated with a special class of Riccati equations. A Riccati equation is a first-order nonlinear ordinary differential equation of the form

$$y'(x) = f_2(x)y^2(x) + f_1(x)y(x) + f_0(x),$$

where f_0, f_1, and f_2 are given functions of x. The differential equation component of the Bellman equation (60)–(61) is a Riccati equation on specific intervals, and the results here facilitate the derivation of its solution; see Appendix B.2 for details. To this end, we consider the following initial value problem:

$$y'(x) \;=\; c_4 y^2(x) \;+\; (c_3 x \;+\; c_2)y(x) \;+\; c_1 x \;+\; c_0, \quad x\in[0,\infty), \tag{99}$$

$$y(0) \;=\; y_0, \tag{100}$$

where $y_0, c_0, c_1, c_2, c_3, c_4 \in \mathbb{R}$ are constants such that $c_4 \neq 0$.

Lemma 2. *There exists a unique solution $y \in C^1[0,\infty)$ to (99)–(100).*

Proof. We first establish the existence of a solution to (99)–(100). This result is included here for completeness, as it previews the closed-form solution derived later. We consider two cases: $c_3 = 0$ and $c_3 \neq 0$. In both cases, the solution can be expressed in terms of special functions; see, e.g., Zaitsev and Polyanin (2002, Sections 1.2.2-2 and 2.1.2-3). If $c_3 = 0$, the solution can be expressed in terms of Airy functions. If $c_3 \neq 0$, the solution can be expressed in terms of confluent hypergeometric functions.

We next establish the uniqueness of a solution to (99)–(100). Observe that (99) can be written in the form $y'(x) = f(x, y(x))$, where $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f(x,y) \coloneqq c_4 y^2 + (c_3 x + c_2) y + c_1 x + c_0, \quad (x,y) \in [0,\infty) \times \mathbb{R}.$$

By the Picard–Lindelöf Theorem, uniqueness follows if f is locally Lipschitz in y, i.e., if f is Lipschitz in y when restricted to the compact set $[0, N] \times [-M, M]$ for arbitrary $N, M > 0$. To see this, note that for $x \in [0, N]$ and $y_1, y_2 \in [-M, M]$,

$$\begin{aligned}
\left|f(x,y_2) - f(x,y_1)\right| &= \left|\left(c_4 y_2^2 + (c_3 x + c_2) y_2\right) - \left(c_4 y_1^2 + (c_3 x + c_2) y_1\right)\right| \\
&\leq \left|c_4\right| \left|y_2^2 - y_1^2\right| + \left|c_3 x + c_2\right| \left|y_2 - y_1\right| \\
&= \Big(\left|c_4\right| \left|y_2 + y_1\right| + \left|c_3 x + c_2\right|\Big) \left|y_2 - y_1\right| \\
&\leq \Big(2M\left|c_4\right| + N\left|c_3\right| + \left|c_2\right|\Big) \left|y_2 - y_1\right| \\
&= L_{N,M} \left|y_2 - y_1\right|,
\end{aligned}$$

where $L_{N,M} \coloneqq 2M\left|c_4\right| + N\left|c_3\right| + \left|c_2\right| < \infty$. Therefore, f is locally Lipschitz in y, and it follows that the solution to (99)–(100) is unique. □

As a preliminary step in solving (99)–(100), we establish the following result, which provides an equivalence between this problem and a second-order initial value problem.

Lemma 3. *For each $y \in C^1[0, \infty)$ satisfying (99)–(100), the function $z(x) \coloneqq \exp\big(-c_4 \int_0^x y(t)\,dt\big)$ for $x \in [0, \infty)$ satisfies*

$$z''(x) - (c_3 x + c_2) z'(x) + c_4 (c_1 x + c_0) z(x) = 0, \quad x \in [0,\infty), \tag{101}$$

$$z(0) = 1, \quad z'(0) = -c_4 y_0. \tag{102}$$

Conversely, for each $z \in C^2[0, \infty)$ satisfying (101)–(102), the function $y(x) \coloneqq -z'(x)/(c_4 z(x))$ for $x \in [0, \infty)$ satisfies (99)–(100).

Proof. Apart from minor notational differences, the proof is identical to that of Lemma 23 in Alwan et al. (2024). □

Next, we discuss the solution to (99)–(100). To simplify the analysis, we assume that $c_3 \geq 0$. This assumption entails no loss of generality, as the Bellman equation satisfies this condition. We proceed by considering two distinct cases: $c_3 = 0$ and $c_3 > 0$.

Case 1: Solution for $\mathbf{c_3 = 0}$. When $c_3 = 0$, (101) simplifies to the following:

$$z''(x) - c_2 z'(x) + c_4 (c_1 x + c_0) z(x) = 0, \quad x \in [0,\infty). \tag{103}$$

Following Ata and Barjesteh (2022, Appendix C), we write the solution to (101) in terms of Airy functions. The Airy functions of the first kind $\mathrm{Ai} : \mathbb{R} \to \mathbb{R}$ and of the second kind $\mathrm{Bi} : \mathbb{R} \to \mathbb{R}$ are defined as

$$\mathrm{Ai}(x) \coloneqq \frac{1}{\pi} \int_0^\infty \cos(\tfrac{1}{3} z^3 + zx)\, dz \quad \text{and} \quad \mathrm{Bi}(x) \coloneqq \frac{1}{\pi} \int_0^\infty \big(\cos(-\tfrac{1}{3} z^3 + zx) + \sin(\tfrac{1}{3} z^3 + zx) \big)\, dz, \quad x \in \mathbb{R}.$$

The next lemma provides a closed-form solution to (99)–(100) when $c_3 = 0$. To state it, define the linear function $u(x) \coloneqq \left[c_2^2 - 4c_4(c_1 x + c_0) \right] / \, 4(c_1 c_4)^{2/3}$ for $x \in \mathbb{R}$, and let C_1 and C_2 be constants given by

$$C_1 \coloneqq \frac{\mathrm{Bi}'(u(0)) - \mathrm{Bi}(u(0))(c_1 c_4)^{-1/3}(c_4 y_0 + \frac{c_2}{2})}{\mathrm{Ai}(u(0))\mathrm{Bi}'(u(0)) - \mathrm{Ai}'(u(0))\mathrm{Bi}(u(0))}, \quad C_2 \coloneqq \frac{-\mathrm{Ai}'(u(0)) + \mathrm{Ai}(u(0))(c_1 c_4)^{-1/3}(c_4 y_0 + \frac{c_2}{2})}{\mathrm{Ai}(u(0))\mathrm{Bi}'(u(0)) - \mathrm{Ai}'(u(0))\mathrm{Bi}(u(0))}.$$

Lemma 4. *Suppose that $c_3 = 0$ and let $z \in C^2[0, \infty)$ be the function defined by*

$$z(x) \coloneqq C_1 \exp\Big(\frac{c_2 x}{2}\Big) \mathrm{Ai}(u(x)) + C_2 \exp\Big(\frac{c_2 x}{2}\Big) \mathrm{Bi}(u(x)), \quad x \in [0, \infty).$$

Then, the function $y(x) \coloneqq -z'(x)/(c_4 z(x))$ for $x \in [0, \infty)$ is the unique solution to (99)–(100).

Proof. Apart from minor notational differences, the proof is identical to that of Lemma EC.2 in Ata and Barjesteh (2022). □

Case 2: Solution for $\mathbf{c_3} > \mathbf{0}$. When $c_3 > 0$, we take a similar approach as above, with the primary difference being that the second-order differential equation in Lemma 3 retains the constant c_3, resulting in a different solution. As shown in Zaitsev and Polyanin (2002, Equation 2.1.2.108) and Abramowitz and Stegun (1965, Chapter 13), the general solution to (101) when $c_3 \neq 0$ is given by

$$z(x) = \exp(Cx)\, u\big(A, \frac{1}{2}, \frac{c_3}{2}(x - B)^2\big), \quad x \in [0, \infty), \tag{104}$$

where $A \coloneqq -c_1^2 c_4^2 / 2c_3^3 + c_2 c_1 c_4 / 2c_3^2 - c_0 c_4 / 2c_3$, $B \coloneqq 2c_1 c_4 / c_3^2 + c_2 / c_3$, and $C \coloneqq c_1 c_4 / c_3$, and the function $x \mapsto u(a, b, x)$ for $x \in [0, \infty)$ is the general solution to the degenerate hypergeometric differential equation

$$x u''(x) + (b - x) u'(x) - a u(x) = 0.$$

When b is not an integer, as is the case in (104), the general solution to this hypergeometric equation can be expressed in terms of Kummer's function, defined as

$$\Phi(a,\, b,\, x) \coloneqq 1 + \sum_{k=1}^\infty \frac{(a)_k}{(b)_k} \frac{x^k}{k!}, \quad x \in [0, \infty),$$

where $(a)_0 \coloneqq 1$ and $(a)_k \coloneqq a(a+1) \cdots (a+k-1)$ for $k \in \mathbb{N}$. Kummer's function is an entire function of x (i.e., an analytic function on the entire real line), except when b is a negative integer; see, e.g., Kummer (1837) and Zaitsev and Polyanin (2002, Section S.2.7) for further details. Using Kummer's function, the general solution to the hypergeometric equation is given by

$$u(x) \;=\; C_1\Phi(a,\,b,\,x) \;+\; C_2x^{1-b}\Phi(a-b+1,\,2-b,\,x), \quad x\in[0,\infty), \tag{105}$$

where C_1 and C_2 are arbitrary constants. The next lemma provides a closed-form solution to (99)–(100) when $c_3 \neq 0$. To state it, let C_1 and C_2 be constants given by

$$C_1 \coloneqq \frac{B(C+c_4y_0)\Phi(A{+}\frac{1}{2},\frac{3}{2},\frac{B^2c_3}{2})-1}{B^2c_3\Phi'(A,\frac{1}{2},\frac{B^2c_3}{2})\Phi(A{+}\frac{1}{2},\frac{3}{2},\frac{B^2c_3}{2})-\Phi(A,\frac{1}{2},\frac{B^2c_3}{2})} \quad \text{and} \quad C_2 \coloneqq \frac{1-C_1\Phi(A,\frac{1}{2},\frac{B^2c_3}{2})}{|B|\sqrt{\frac{c_3}{2}}\,\Phi(A{+}\frac{1}{2},\frac{3}{2},\frac{B^2c_3}{2})}. \tag{106}$$

Lemma 5. *Suppose that $c_3 > 0$ and let $z \in C^2[0,\infty)$ be the function defined by*

$$z(x) \;\coloneqq\; \exp(Cx)\left(C_1\Phi\big(A,\frac{1}{2},\frac{c_3}{2}(x-B)^2\big) \;+\; C_2\,x^{1/2}\,\Phi\big(A+\frac{1}{2},\frac{3}{2},\frac{c_3}{2}(x-B)^2\big)\right), \quad x\in[0,\infty).$$

Then, the function $y(x) \coloneqq -z'(x)/(c_4z(x))$ for $x\in[0,\infty)$ is the unique solution to (99)–(100).

Proof. It follows from Lemma 3, along with (104)–(105) and the surrounding discussion, that y satisfies (99). Moreover, by the product rule and the chain rule for differentiation, for $x\in[0,\infty)$ it follows that

$$\begin{aligned} z'(x) \;=\; C\exp(Cx)\,z(x) \;+\; \exp(Cx)\bigg(&C_1c_3(x-B)\Phi'\big(A,\frac{1}{2},\frac{c_3}{2}(x-B)^2\big) \\ &+ C_2\sqrt{\tfrac{c_3}{2}}\,\mathrm{sgn}(x-B)\Big(\Phi\big(A+\frac{1}{2},\frac{3}{2},\frac{c_3}{2}(x-B)^2\big) \;+\; c_3(x-B)^2\Phi'\big(A+\frac{1}{2},\frac{3}{2},\frac{c_3}{2}(x-B)^2\big)\Big)\bigg). \end{aligned}$$

Using the expressions for C_1 and C_2 in (106), it is easily verified that $z(0) = 1$ and $z'(0) = -c_4y_0$. We conclude that y satisfies (100). Finally, uniqueness follows from Lemma 2. □

B.2 Existence of a Solution to the Bellman Equation

In this section, we analyze two IVPs that are closely related to the Bellman equation (60)–(61) introduced in Section 6.2. As a preliminary step, we first establish key properties of the effective state cost function in Section B.2.1 and then present additional auxiliary results in Section B.2.2 to facilitate the analysis of the IVPs. Finally, in Section B.2.3, we introduce the IVPs and leverage the results from Sections B.2.1 and B.2.2 to prove the existence of a unique solution to the Bellman equation.

B.2.1 Characterization of the Effective State Cost Function

To facilitate the analysis to follow, we recall key definitions from Section 6.2. First, the effective state cost function $\phi : \mathbb{R}^2 \to \mathbb{R}$, originally defined in (59), is restated here for convenience:

$$\phi(w,y) \;\coloneqq \min_{z\in\mathcal{A}(w)} \varphi(z,y), \quad w,y\in\mathbb{R},$$

where $\mathcal{A}(w)$ denotes the set of admissible workload distribution vectors at workload level w, as defined in (45). Second, the function $\varphi : \mathbb{R}^K \times \mathbb{R} \to \mathbb{R}$, originally defined in (55), is restated here for convenience:

$$\varphi(z, y) \coloneqq \sum_{k \in \mathcal{S}_{\mathrm{w}}} \ell_k \big(d_k - m_k y\big) z_k^+ + \sum_{k \in \mathcal{S}} \upsilon_k \big(z_k - \lambda_k^\star \delta_k\big) + \sum_{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} h_k z_k^-, \quad (z, y) \in \mathbb{R}^K \times \mathbb{R},$$

where the cost function υ_k for $k \in \mathcal{S}$ is given by (13). Third, the cost terms $\hat{\alpha}_k$ and $\hat{\beta}_k$, originally defined in (66), are restated here for convenience:

$$\hat{\alpha}_k(y) \coloneqq \begin{cases} \alpha_k + \ell_k(d_k - m_k y), & k \in \mathcal{S}_{\mathrm{w}}, \\ \alpha_k, & k \in \mathcal{S}_{\mathrm{o}}, \end{cases} \qquad \text{and} \qquad \hat{\beta}_k \coloneqq \begin{cases} \beta_k, & k \in \mathcal{S}_{\mathrm{o}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}, \\ h_k, & k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}. \end{cases}$$

The corresponding cost functions $\hat{\upsilon}_k : \mathbb{R} \to \mathbb{R}$ for $k \in \mathcal{S}$ (comprising tardiness, earliness, holding, and abandonment costs), originally defined in (67), are restated here for convenience:

$$\hat{\upsilon}_k(x, y) \coloneqq \begin{cases} \hat{\alpha}_k(y)\, x, & x \geq 0, \\ -\hat{\beta}_k\, x, & x < 0. \end{cases}$$

Finally, the function φ, which was rewritten in (68), is restated it here for convenience:

$$\varphi(z, y) = \sum_{k \in \mathcal{S}_{\mathrm{o}}} \hat{\upsilon}_k(z_k - \lambda_k^\star \delta_k, y) + \sum_{k \in \mathcal{S}_{\mathrm{w}}} \hat{\upsilon}_k(z_k, y), \quad (z, y) \in \mathbb{R}^K \times \mathbb{R}.$$

The next two results establish that as long as $y \leq \kappa$, the workload cannot be distributed in a way that results in a negative effective state cost.[12] The restriction $y \leq \kappa$ also allows us to derive a closed-form expression for ϕ, which facilitates the analysis in Section B.2.3. Nevertheless, as noted in the footnote, this restriction is harmless for our purposes.

Lemma 6. *For all $k \in \mathcal{S}$ and $y \in (-\infty, \kappa]$, we have that $\hat{\alpha}_k(y) > 0$ and $\hat{\beta}_k \geq 0$, with $\hat{\beta}_k > 0$ for all $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$.*

Proof. By (44) and Assumption 2, we have $\kappa \leq d_k/m_k$ for all $k \in \mathcal{S}_{\mathrm{w}}$, implying that $d_k - m_k y \geq 0$ for all such k. Also recall that $\alpha_k > 0$ for all $k \in \mathcal{S}$, $\ell_k \geq 0$ for all $k \in \mathcal{S}_{\mathrm{w}}$, $\beta_k = 0$ for $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} \cup \mathcal{S}^{\mathrm{MTS}}$, $\beta_k > 0$ for $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}$, and $h_k > 0$ for $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ (see Sections 3.4 and 4.2). Substituting these inequalities into the definitions of $\hat{\alpha}_k(y)$ and $\hat{\beta}_k$ in (66), it follows that $\hat{\alpha}_k(y) > 0$ for all $k \in \mathcal{S}$ and $y \in (-\infty, \kappa]$, $\hat{\beta}_k > 0$ for all $k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$, and $\hat{\beta}_k = 0$ for all $k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}$. $\square$

Corollary 3. *For all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$, we have that $\phi(w, y) \geq 0$.*

Proof. By (67)–(68) and Lemma 6, we have $\varphi(z, y) \geq 0$ for all $z \in \mathbb{R}^K$ and $y \in (-\infty, \kappa]$. It then follows from (59) that $\phi(w, y) \geq 0$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$. $\square$

[12] The restriction $y \leq \kappa$ is needed because if $y > \kappa$, there could exist a class $k \in \mathcal{S}_{\mathrm{w}}$ such that $d_k - m_k y < 0$. In this case, one could increase z_k^+ without bound, leading to an unbounded state cost from below. This restriction is innocuous, as the optimal value function (as we will show later) satisfies $v(w) \leq \kappa$ for $w \in [l, u]$.

We next provide a closed-form characterization of the effective state cost function ϕ when $y \in (-\infty, \kappa]$ by means of identifying a minimizer on the right-hand side of (59). Since here we focus solely on the mathematical results, we encourage the reader to refer to Section 6.3—in particular, Figures 2 and 3—for a more detailed intuitive explanation of the mathematical expressions that follow. To that end, recall the constants w_0 and w_1, originally defined in (62), which are restated here for convenience:

$$w_0 \coloneqq \sum_{k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}} m_k \lambda_k^\star \delta_k \quad \text{and} \quad w_1 \coloneqq \sum_{k \in \mathcal{S}_{\mathrm{o}}} m_k \lambda_k^\star \delta_k.$$

These constants are introduced because the structure of the effective state cost function and its minimizer differs across the three intervals $(-\infty, w_0)$, $[w_0, w_1]$, and (w_1, ∞). Before stating the results, we define the (not necessarily unique) bijective mapping $i^\star : \{1, \dots, |\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}| + |\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}|\} \to \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ that orders the online MTO and walk-in MTS products in increasing order of $\hat{\beta}_k / m_k$, so that

$$\hat{\beta}_{i^\star(k)} / m_{i^\star(k)} \;\le\; \hat{\beta}_{i^\star(k+1)} / m_{i^\star(k+1)} \quad \text{for} \quad k \,=\, 1\,, \,\dots\,, \,|\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}| + |\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}| - 1.$$

We then define $I \coloneqq \min\{1 \le k \le |\mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}| + |\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}| : i^\star(k) \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}\}$, which corresponds to the index of the cheapest walk-in MTS product according to the ordering above. In particular, $i^\star(I) \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}$ is the cheapest walk-in MTS product according to the ordering above, while products $i^\star(j) \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}$ for $j = 1, \dots, I-1$ are the online MTO products that are cheaper than $i^\star(I)$ in the same sense. We now define the product class $k^\star(w, y) \in \mathcal{S}$ for $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$ as follows:

$$k^\star(w,y) \;\coloneqq\; \begin{cases} \max\big\{1 \le k \le I \,:\, \sum_{j=1}^{k-1} m_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)} \le w_0 - w \big\}, & w \in (-\infty, w_0), \\ \min\big\{k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}} \,:\, \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}} : j \le k} m_k \lambda_k^\star \delta_k > w - w_0 \big\}, & w \in [w_0, w_1], \\ \operatorname*{argmin}_{k \in \mathcal{S}} \big\{\hat{\alpha}_k(y)/m_k\big\}, & w \in (w_1, \infty). \end{cases} \tag{107}$$

Notice that $k^\star(w, y)$ depends only on w when $w \in (-\infty, w_1]$ and depends only on y when $w \in (w_1, \infty)$.[13] It is straightforward to verify that we can equivalently, and more conveniently, express $k^\star(w, y)$ for $w \in (-\infty, w_0)$ and $y \in (-\infty, \kappa]$ as follows:

$$k^\star(w,y) \;=\; \begin{cases} I - k, & w \in [\,\tau^-_{I-k}\,,\, \tau^-_{I-k-1}), \quad k \,=\, 1, \,\dots, \,I-1, \\ I, & w \in (-\infty\,,\, \tau^-_{I-1}), \end{cases} \tag{108}$$

[13]Equation (107) provides a single, unified definition of the product class $k^\star(w, y)$ for all workload levels, rather than defining it separately for different workload intervals. However, when the context permits, we may drop y (resp., w) and write $k^\star(w)$ (resp., $k^\star(y)$) in favor of $k^\star(w, y)$ when $w \le w_1$ (resp., $w > w_1$).

where the interval endpoints τ_k^- are given as follows:

$$\tau_k^- \coloneqq \begin{cases} w_0, & k = 0, \\ w_0 - \displaystyle\sum_{j=1}^{k} m_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, & k = 1, \ldots, I-1. \end{cases} \tag{109}$$

A more detailed description of $k^\star(w, y)$ for $w \in (w_1, \infty)$ and $y \in (-\infty, \kappa]$ is provided in Section B.2.2.

With the above notation established, we next identify a minimizer of the right-hand side of (59) when $y \in (-\infty, \kappa]$. Define the workload configuration $\mathcal{Z} : \mathbb{R} \times (-\infty, \kappa] \to \mathbb{R}^K$ as follows: For $w \in (-\infty, w_0)$, the workload configuration is given by

$$\mathcal{Z}_k(w, y) \coloneqq \begin{cases} \lambda^\star_k \delta_k, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} \setminus \{i^\star(1), \ldots, i^\star(k^\star(w, y))\}, \\ \lambda^\star_k \delta_k - (\tau^-_{k^\star(w,y)-1} - w)/m_k, & k = i^\star(k^\star(w, y)), \\ 0, & \text{otherwise.} \end{cases} \tag{110}$$

For $w \in [w_0, w_1]$, the workload configuration is given by

$$\mathcal{Z}_k(w, y) \coloneqq \begin{cases} \lambda^\star_k \delta_k, & k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}} \cup \{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}} : j < k^\star(w, y)\}, \\ (w - w_0 - \displaystyle\sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}} : j < k^\star(w,y)} m_j \lambda^\star_j \delta_j)/m_k, & k = k^\star(w, y), \\ 0, & \text{otherwise.} \end{cases} \tag{111}$$

For $w \in (w_1, \infty)$, the workload configuration is given by

$$\mathcal{Z}_k(w, y) \coloneqq \begin{cases} \lambda^\star_k \delta_k + (w - w_1)/m_k, & k = k^\star(w, y), \\ \lambda^\star_k \delta_k, & \text{otherwise.} \end{cases} \tag{112}$$

The following lemma shows that the workload configuration defined above is an optimal workload configuration, as it minimizes the effective state costs.

Lemma 7. *The workload configuration $\mathcal{Z}$ defined in (110)–(112) satisfies $\mathcal{Z}(w, y) \in \operatorname{argmin}_{z \in \mathcal{A}(w)} \varphi(z, y)$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$, i.e., is an optimal workload configuration. Moreover, the optimal workload configuration need not be unique in general.*

Proof. We must show that $\mathcal{Z}(w, y) \in \operatorname{argmin}_{z \in \mathcal{A}(w)} \varphi(z, y)$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$. To do so, we first verify that $\mathcal{Z}(w, y) \in \mathcal{A}(w)$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$. For example, when $w \in (w_1, \infty)$, we have

$$m' \mathcal{Z}(w, y) = \sum_{k \in \mathcal{S}} \lambda^\star_k \delta_k m_k + (w - w_1) = w,$$

where the second equality follows from the fact that $w_1 = \sum_{k \in \mathcal{S}_{\mathrm{o}}} \lambda^\star_k \delta_k m_k = \sum_{k \in \mathcal{S}} \lambda^\star_k \delta_k m_k$, which holds since $\delta_k = 0$ for $k \in \mathcal{S}_{\mathrm{w}}$. The cases of $w \in (-\infty, w_0)$ and $w \in [w_0, w_1]$ follow similarly and are therefore omitted.

To complete the proof, we must show that $\varphi(z, y) \geq \varphi(\mathcal{Z}(w, y), y)$ for all $w \in \mathbb{R}$, $y \in (-\infty, \kappa]$, and

$z \in \mathcal{A}(w)$. We establish this inequality by considering w over the intervals $(-\infty, w_0)$, $[w_0, w_1]$, and (w_1, ∞). While we omit the full mathematical details, we provide sufficient explanation to guide the reader through the argument.

First, consider the case where $w \in [w_0, w_1]$. In this case, the workload configuration $\mathcal{Z}(w, y)$ allocates a workload of w_0 to the online MTO products without incurring earliness or tardiness costs. The remaining workload, $w - w_0$, is held in the online MTS products while ensuring that $z_k^\star(w, y) \leq \lambda_k^\star \delta_k$ for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$. Consequently, no tardiness costs are incurred for the online MTO products. No workload is held in the walk-in classes, which ensures their earliness, tardiness, and abandonment costs are zero. We conclude that $\varphi(\mathcal{Z}(w, y), y) = 0$, and so the desired result follows from Corollary 3. Finally, since there are uncountably many ways to distribute workload among the online MTS products without incurring tardiness costs, the optimal workload configuration need not be unique.

Second, consider the case where $w \in (w_1, \infty)$. In this case, the workload configuration $z_k^\star(w, y)$ holds a workload of w_1 in the online classes without incurring earliness or tardiness costs. The remaining workload, $w - w_1$, is then held in the cheapest manner, i.e., in the product class with the lowest (effective) tardiness and abandonment cost per unit of work.

Finally, consider the case where $w \in (-\infty, w_0)$. In this case, the workload configuration $z_k^\star(w, y)$ initially allocates a workload of w_0 to the online MTO products, with all other products receiving zero workload. If left unchanged, this allocation would result in zero earliness, tardiness, holding, and abandonment costs across all product classes. However, since $w_0 - w > 0$, the workload configuration subtracts this excess workload in the cheapest manner by decreasing the workload of the online MTO and walk-in MTS products in increasing order of earliness or holding cost per unit of work. Meanwhile, the workload of the walk-in MTO and online MTS products remains zero. □

The following corollary provides a closed-form expression for the effective state cost function.

Corollary 4. *For $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$, the effective state cost function $\phi(w, y)$ is given by*

$$\phi(w, y) = \begin{cases} \hat{\beta}_{i^\star(k^\star(w,y))}(\tau^-_{k^\star(w,y)-1} - w)/m_{i^\star(k^\star(w,y))} + \sum_{j=1}^{k^\star(w,y)-1} \hat{\beta}_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, & w \in (-\infty, w_0), \\ 0, & w \in [w_0, w_1], \\ \hat{\alpha}_{k^\star(w,y)}(y)(w - w_1)/m_{k^\star(w,y)}, & w \in (w_1, \infty). \end{cases}$$

Proof. By Lemma 7, it follows that $\phi(w, y) = \varphi(\mathcal{Z}(w, y), y)$ for all $w \in \mathbb{R}$ and $y \in (-\infty, \kappa]$. Then, substituting (110)–(112) into (55) and (59) and simplifying the terms yields the desired result. □

The following lemma establishes some structural properties of the effective state cost function, which will be used in Section B.2.3.

Lemma 8. *For each $y \in (-\infty, \kappa]$, the mapping $w \mapsto \phi(w, y)$ has the following properties:*

(a) It is continuous everywhere and differentiable almost everywhere with respect to Lebesgue measure.

(b) It is strictly decreasing on $(-\infty, w_0)$ with $\phi(w, y) \to \infty$ as $w \to -\infty$, constant on $[w_0, w_1]$ with $\phi(w, y) \equiv 0$, and strictly increasing on (w_1, ∞) with $\phi(w, y) \to \infty$ as $w \to \infty$.

Proof. Fix $y \in (-\infty, \kappa]$. We first establish part (a). It follows from (108)–(109) and Corollary 4 that the mapping $w \mapsto \phi(w, y)$ is piecewise linear over its entire domain. Thus, to prove continuity, it suffices to show that it is continuous at its breakpoints, i.e., the points where the derivative changes. It follows from (108) and Corollary 4 that the breakpoints are $\tau_{I-1}^-, \ldots, \tau_1^-, \tau_0^-$, and w_1. It is straightforward to verify that $w \mapsto \phi(w, y)$ is left-continuous on its entire domain and that its left and right limits agree at each breakpoint. Therefore, $w \mapsto \phi(w, y)$ is continuous. Moreover, since it is piecewise linear, it is differentiable except at a set of Lebesgue measure zero, completing the proof of part (a).

We next establish part (b). On the interval $(-\infty, w_0)$, since $\hat{\beta}_{i^\star(k^\star(w,y))} > 0$ (by Lemma 6) and $i^\star(k^\star(w, y)) \in \mathcal{S}_{\text{o}}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$ (by the definition of the mapping $i^\star$), it follows from Corollary 4 and part (a) that $w \mapsto \phi(w, y)$ is a continuous and piecewise linear function with negative slopes. Consequently, $w \mapsto \phi(w, y)$ is strictly decreasing on $(-\infty, w_0)$ with $\phi(w, y) \to \infty$ as $w \to -\infty$. On the interval $[w_0, w_1]$, it follows from Corollary 4 that $\phi(w, y) = 0$ for all $w \in [w_0, w_1]$, as desired. On the interval (w_1, ∞), since $\hat{\alpha}_{k^\star(w,y)}(y) > 0$ (by Lemma 6) and $w \mapsto k^\star(w, y)$ remains constant over the entire interval (by (107)), it follows from Corollary 4 and part (a) that $w \mapsto \phi(w, y)$ is a linear function with a constant positive slope. Consequently, $w \mapsto \phi(w, y)$ is strictly increasing on (w_1, ∞) with $\phi(w, y) \to \infty$ as $w \to \infty$. □

B.2.2 Characterization of the Product Class $k^\star(w, y)$ for $w \in (w_1, \infty)$

In this section, we further analyze the product class $k^\star(w, y)$ defined in (107), focusing specifically on the case where $w \in (w_1, \infty)$ and $y \in (-\infty, \kappa]$. An explicit characterization of this class will be useful for solving the Bellman equation later. To facilitate the analysis, define the function $\mathbf{A} : (-\infty, \kappa] \to \mathbb{R}$ as follows:

$$\mathbf{A}(y) \coloneqq \min_{k \in \mathcal{S}} \{\hat{\alpha}_k(y)/m_k\}, \quad y \in (-\infty, \kappa]. \tag{113}$$

Moreover, we introduce the set $\mathcal{S}^0 \coloneqq \mathcal{S}_{\text{o}} \cup \{k \in \mathcal{S}_{\text{w}} : \ell_k = 0\}$, which consists of all products with a zero abandonment rate. The following result provides some useful structural properties of $\mathbf{A}$.

Lemma 9. *The function $\mathbf{A} : (-\infty, \kappa] \to \mathbb{R}$ is strictly positive, nonincreasing, continuous, concave, and non-differentiable at most at finitely many points. Furthermore, it is either constant over its entire domain or remains constant up to its first point of non-differentiability, after which it has a strictly negative derivative at all points where it is differentiable.*

Proof. We begin by establishing that $\mathbf{A}$ is strictly positive, nonincreasing, continuous, concave, and non-differentiable at most at finitely many points. First, by Lemma 6, we have that $\mathbf{A}(y) > 0$ for $y \in (-\infty, \kappa]$, i.e., $\mathbf{A}$ is strictly positive. Second, by (66), the mapping $y \mapsto \hat{\alpha}_k(y)$ is nonincreasing and continuous for each $k \in \mathcal{S}$. Since the pointwise infimum of nonincreasing and continuous functions is nonincreasing and continuous, it follows that $\mathbf{A}$ is nonincreasing and continuous. Third, since the mappings $y \mapsto \hat{\alpha}_k(y)$ for $k \in \mathcal{S}$ are affine and the pointwise infimum of affine functions is concave (see, e.g., Boyd and Vandenberghe (2004, Chapter 3)), it follows that $\mathbf{A}$ is concave. Finally, since $\mathbf{A}$ is the pointwise minimum of (a finite number of) affine functions, it can only be non-differentiable at points where two of these functions intersect. Consider $k_1, k_2 \in \mathcal{S}$ such that $\ell_{k_1} < \ell_{k_2}$, and suppose that their corresponding affine functions intersect at some $y' \in (-\infty, \kappa]$, i.e., $\hat{\alpha}_{k_1}(y')/m_{k_1} = \hat{\alpha}_{k_2}(y')/m_{k_2}$. Since $\ell_{k_1} < \ell_{k_2}$, it follows that $\hat{\alpha}_{k_1}(y)/m_{k_1} < \hat{\alpha}_{k_2}(y)/m_{k_2}$ for all $y < y'$. This implies that each affine function is either never equal to the pointwise minimum or it is equal to the pointwise minimum on a convex interval. Since there are only finitely many such functions, $\mathbf{A}$ is non-differentiable at most at finitely many points.[14]

We now show that $\mathbf{A}$ is either constant over its entire domain or remains constant up to some point, after which it has a strictly negative derivative except at finitely many points. We do so by considering two exhaustive cases.

Case 1. Assume that there exists $k_0 \in \mathcal{S}^0$ such that $\alpha_{k_0}/m_{k_0} \leq \hat{\alpha}_k(\kappa)/m_k$ for all $k \in \mathcal{S}$. It follows from (66) and Lemma 6 that the mapping $y \mapsto \hat{\alpha}_k(y)$ is nonincreasing on $(-\infty, \kappa]$ for all $k \in \mathcal{S}$. Therefore, $\alpha_{k_0}/m_{k_0} \leq \hat{\alpha}_k(y)/m_k$ for all $k \in \mathcal{S}$ and $y \leq \kappa$. By the definition of $\mathbf{A}$ in (113), it then follows that $\mathbf{A}(y) = \alpha_{k_0}/m_{k_0}$ for all $y \leq \kappa$, which implies that $\mathbf{A}$ is a constant function.

Case 2. Suppose that the assumption in the previous case does not hold, i.e., there exists $k_0 \in \{k \in \mathcal{S}_{\mathrm{w}} : \ell_k > 0\}$ such that $\hat{\alpha}_{k_0}(y)/m_{k_0} < \hat{\alpha}_k(y)/m_k$ for all $k \in \mathcal{S}^0$ in a neighborhood to the left of κ. It follows that $\mathbf{A}'(y) = -\ell_{k_0} < 0$ for all $y \in (y_1, \kappa)$, where y_1 is the largest point such that $\hat{\alpha}_{k_0}(y_1)/m_{k_0} = \hat{\alpha}_{k_1}(y_1)/m_{k_1}$ for some $k_1 \in \mathcal{S}\backslash\{k_0\}$.[15] But then $-\ell_{k_0} < -\ell_{k_1} \leq 0$, $\lim_{y \nearrow y_1} \mathbf{A}'(y) = -\ell_{k_1}$, and $\lim_{y \searrow y_1} \mathbf{A}'(y) = -\ell_{k_0}$, implying that $\mathbf{A}$ is not differentiable at y_1. Continuing iteratively, we find the largest point y' such that there exists $k' \in \mathcal{S}^0$ with $\alpha_{k'}/m_{k'} \leq \hat{\alpha}_k(y)/m_k$ for all $k \in \mathcal{S}$ and $y \leq y'$. We conclude that $\mathbf{A}$ is constant up to y', after which it has a strictly negative derivative except at finitely many points where it is non-differentiable. The precise mathematical details are straightforward, though somewhat tedious, and are therefore omitted. $\square$

By Lemma 9, the number of points where $\mathbf{A}$ is non-differentiable is finite; denote this number by $b \in \mathbb{N}$.

[14] Since only products in $\mathcal{S}^0$ can achieve the minimum in (113), there are at most $K - |\mathcal{S}^0| - \mathbf{1}_{\{|\mathcal{S}^0|=0\}}$ points where $\mathbf{A}$ is non-differentiable. However, this detail is not essential for our purposes.

[15] Since $\hat{\alpha}_{k_0}(y)/m_{k_0} \to \infty$ as $y \to -\infty$, there exists $\bar{k}_1 \in \mathcal{S}^0$ and $\bar{y}_1 < \kappa$ such that $\hat{\alpha}_{\bar{k}_1}(y)/m_{\bar{k}_1} < \hat{\alpha}_{k_0}(y)/m_{k_0}$ for $y < \bar{y}_1$, where $\hat{\alpha}_{\bar{k}_1}(\bar{y}_1)/m_{\bar{k}_1} = \hat{\alpha}_{k_0}(\bar{y}_1)/m_{k_0}$. This proves that y_1 exists.

Setting $y_0 \coloneqq \kappa$, we define the products k_j and points y_j iteratively as follows:[16]

$$k_j \coloneqq \operatorname*{argmin}_{k\in\mathcal{S}\setminus\{k_0,\dots,k_{j-1}\}} \{\hat{\alpha}_k(y_j)/m_k\}, \quad j=0,1,\dots,b, \tag{114}$$

$$y_j \coloneqq \sup\{y<y_{j-1} : \hat{\alpha}_k(y)/m_k < \hat{\alpha}_{k_j}(y)/m_{k_j} \text{ for some } k\in\mathcal{S}\}, \quad j=1,\dots,b. \tag{115}$$

It follows that $k_0,\dots,k_{b-1}\in\{k\in\mathcal{S}_{\mathrm{w}} : \ell_k>0\}$ and $k_b\in\mathcal{S}^0$. (Note that the points y_j given by (115) are precisely those where the derivative of $\mathbf{A}$ changes.) We now provide a more explicit characterization of the product class $k^\star(w,y)$ for $w\in(w_1,\infty)$ and $y\in(-\infty,\kappa]$ and the function $\mathbf{A}$. This characterization will be useful in the analysis of the IVPs in Section B.2.3.

Corollary 5. *The product class $k^\star(w,y)$ for $w\in(w_1,\infty)$ and $y\in(-\infty,\kappa]$, as well as the function $\mathbf{A} : (-\infty,\kappa]\to\mathbb{R}$, can be equivalently expressed as follows, respectively:*

$$k^\star(w,y) = \begin{cases} k_j, & y\in(y_{j+1},y_j], \quad j=0,1,\dots,b-1,\\ k_b, & y\in(-\infty,y_b],\end{cases}$$

$$\mathbf{A}(y) = \begin{cases} \hat{\alpha}_{k_j}(y)/m_{k_j}, & y\in(y_{j+1},y_j], \quad j=0,1,\dots,b-1,\\ \alpha_{k_b}/m_{k_b}, & y\in(-\infty,y_b].\end{cases}$$

Proof. The product classes k_j are chosen iteratively as the minimizers at each breakpoint y_j, where y_j is defined as the largest point where the minimizer changes; see (114)–(115). Thus, on each interval $(y_{j+1},y_j]$, the same product k_j attains the minimum. By Lemma 9, $\mathbf{A}$ has finitely many breakpoints. Therefore, the iterative construction of k_j and y_j terminates in a finite number of steps. Substituting this representation into (107) and (113) yields the desired result. □

B.2.3 Two Initial Value Problems Related to the Bellman Equation

In this section, we analyze two IVPs that are closely related to the Bellman equation (60)–(61). The first IVP constructs a solution on the interval $(-\infty,w_1]$, while the second IVP does so on the interval $[w_1,\infty)$. By pasting together these solutions, we prove the existence of a solution to the Bellman equation, thereby proving Theorem 1. Throughout the analysis, we attach the superscripts "−" and "+" to the various quantities of interest associated with the first and second IVPs, respectively.

[16] Whenever (114) does not have a unique minimizer, we choose the product k_j for which $\hat{\alpha}_{k_j}(y)/m_{k_j} < \hat{\alpha}_k(y)/m_k$ for all $k\in\mathcal{S}\setminus\{k_0,\dots,k_{j-1},k_j\}$ and all $y\in(y_j-\epsilon,y_j)$ for some sufficiently small $\epsilon>0$. (If such a product k_j does not exist, there exist $n_j\geq 2$ products $k_j(1),\dots,k_j(n_j)$ such that $\hat{\alpha}_{k_j(1)}(y)/m_{k_j(1)} = \cdots = \hat{\alpha}_{k_j(n_j)}(y)/m_{k_j(n_j)}$ for all $y\in\mathbb{R}$ and $\hat{\alpha}_{k_j(1)}(y)/m_{k_j(1)} < \hat{\alpha}_k(y)/m_k$ for all $k\in\mathcal{S}\setminus\{k_0,\dots,k_{j-1},k_j(1),\dots,k_j(n_j)\}$ and all $y\in(y_j-\epsilon,y_j)$ for some sufficiently small $\epsilon>0$. In this case, we arbitrarily choose one of the $k_j(i)$ as the minimizer in (114).) This ensures that $y_j\neq y_{j-1}$, guaranteeing that all breakpoints of $\mathbf{A}$ are captured in (115).

B.2.3.1 IVP on $(-\infty, w_1]$**.** For each $\gamma \geq 0$, consider the following IVP:

$$v'(w) = \frac{m'H^{-1}m}{2\sigma^2}\, v^2(w) - \frac{2\mu}{\sigma^2}\, v(w) - \frac{2}{\sigma^2}\, \phi(w, v(w)) + \frac{2\gamma}{\sigma^2}, \quad w \in [l_\gamma, w_1], \tag{116}$$

$$v(l_\gamma) = 0, \tag{117}$$

where the lower barrier $l_\gamma \in \mathbb{R}$ is given by[17]

$$l_\gamma \coloneqq \sup\big\{ l \leq w_0 \,:\, \phi(l, 0) = \gamma \big\}. \tag{118}$$

It follows from (118) that any solution $v \in C^1[l_\gamma, w_1]$ to (116)–(117) satisfies $v'(l_\gamma) = 0$. Therefore, we interpret the solution to this IVP as the value function associated with a barrier policy (with a lower barrier at l_γ) that achieves a long-run average cost of γ and satisfies the smooth pasting condition at the lower barrier; see Proposition 3.

We next prove the existence of a closed-form solution to the IVP (116)–(117) when γ is not too large.[18] In particular, the solution is constructed iteratively over subintervals where the effective state cost function ϕ is linear, allowing us to leverage the results from Section B.1 to derive a closed-form solution. To that end, we partition the interval $[l_\gamma, w_1]$ as follows:

$$[l_\gamma\,,\, w_1] = [l_\gamma\,,\, \tau^-_{k^\star(l_\gamma)-1}) \cup \bigcup_{k=1}^{k^\star(l_\gamma)-1} [\tau^-_{k^\star(l_\gamma)-k}\,,\, \tau^-_{k^\star(l_\gamma)-k-1}) \cup [w_0\,,\, w_1], \tag{119}$$

where $\tau^-_{k^\star(l_\gamma)} \coloneqq l_\gamma$ for notational convenience. (Note that since $k^\star(w, y)$ only depends on w (and not y) for $w \in (-\infty, w_1]$, we write $k^\star(l_\gamma)$ for simplicity; see (107).) We proceed in four main steps. In Step 1, we find the unique solution on $[l_\gamma, \tau^-_{k^\star(l_\gamma)-1}]$. In Step 2, we use the value of the solution from Step 1 at $\tau^-_{k^\star(l_\gamma)-1}$ as the initial condition and find the unique solution on $[\tau^-_{k^\star(l_\gamma)-1}, \tau^-_{k^\star(l_\gamma)-2}]$. This process continues iteratively, where the solution from the previous interval provides the initial condition for the next, until we obtain unique solutions on $[\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}]$ for $k = 1, \ldots, k^\star(l_\gamma) - 1$. In Step 3, we use the value of the last solution from Step 2 at w_0 as the initial condition and find the unique solution on $[w_0, w_1]$. In Step 4, we combine the solutions from Steps 1–3 to obtain a continuously differentiable function on $[l_\gamma, w_1]$. After constructing this function, we establish its key properties and show that for γ belonging to a certain compact set, it uniquely solves the IVP (116)–(117).

Step 1: Solution on $\big[l_\gamma, \tau^-_{k^\star(l_\gamma)-1}\big)$**.** We construct a solution on the interval $[l_\gamma, \tau^-_{k^\star(l_\gamma)-1})$ by considering

[17] The existence of l_γ follows from Lemma 8, since the mapping $w \mapsto \phi(w, 0)$ is continuous and strictly decreasing on $(-\infty, w_0]$, with $\phi(w_0, 0) = 0$ and $\phi(w, 0) \to -\infty$ as $w \to -\infty$.

[18] This statement will be made precise later. The heart of the matter is that as long as γ remains within an appropriate range, the solution to the IVP (116)–(117) will be (pointwise) bounded above by κ, allowing us to apply the closed-form expression for $\phi(w, y)$ when $y \in (-\infty, \kappa]$.

the following IVP:

$$v'(w) \;=\; \frac{m'H^{-1}m}{2\sigma^2}\, v^2(w) \;-\; \frac{2\mu}{\sigma^2}\, v(w) \;-\; \frac{2}{\sigma^2}\, \phi^-_{k^\star(l_\gamma)}(w) \;+\; \frac{2\gamma}{\sigma^2}, \quad w \in [l_\gamma, \infty), \tag{120}$$

$$v(l_\gamma) \;=\; 0, \tag{121}$$

where

$$\phi^-_{k^\star(l_\gamma)}(w) \;\coloneqq\; \frac{\hat\beta_{i^\star(k^\star(l_\gamma))}}{m_{i^\star(k^\star(l_\gamma))}} \left(\tau^-_{k^\star(l_\gamma)-1} - w\right) \;+\; \sum_{j=1}^{k^\star(l_\gamma)-1} \hat\beta_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, \quad w \in [l_\gamma, \infty), \tag{122}$$

is the linear function that coincides with $\phi(w,y)$ for $w \in [l_\gamma, \tau^-_{k^\star(l_\gamma)-1}]$ and $y \in (-\infty, \kappa]$. Since $\phi^-_{k^\star(l_\gamma)}$ is a linear function, we apply the results from Section B.1 to obtain a solution to (120)–(121). To be specific, consider the Riccati equation (99)–(100) with the following constants:

$$c_0 \;=\; -\frac{2}{\sigma^2}\left[\frac{\hat\beta_{i^\star(k^\star(l_\gamma))}}{m_{i^\star(k^\star(l_\gamma))}} \left(\tau^-_{k^\star(l_\gamma)-1} + l_\gamma\right) \;+\; \sum_{j=1}^{k^\star(l_\gamma)-1} \hat\beta_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)} - \gamma\right],$$

$$c_1 \;=\; \frac{2}{\sigma^2}\frac{\hat\beta_{i^\star(k^\star(l_\gamma))}}{m_{i^\star(k^\star(l_\gamma))}}, \quad c_2 \;=\; -\frac{2\mu}{\sigma^2}, \quad c_3 \;=\; 0, \quad c_4 \;=\; \frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 \;=\; 0.$$

By Lemma 4, there exists a unique solution $y^-_{\gamma,\,k^\star(l_\gamma)} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat v^-_{\gamma,\,k^\star(l_\gamma)} \in C^1[l_\gamma,\infty)$ given by

$$\hat v^-_{\gamma,\,k^\star(l_\gamma)}(w) \;\coloneqq\; y^-_{\gamma,\,k^\star(l_\gamma)}(w - l_\gamma), \quad w \in [l_\gamma, \infty) \tag{123}$$

is the unique solution to (120)–(121).

Step 2: Solution on $[\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1})$ **for** $k \,=\, 1, \ldots, k^\star(l_\gamma) - 1$**.** Given the solution $\hat v^-_{\gamma,\,k^\star(l_\gamma)-k+1}$ on the interval $[\tau^-_{k^\star(l_\gamma)-k+1}, \tau^-_{k^\star(l_\gamma)-k})$ for $k \,=\, 1, \ldots, k^\star(l_\gamma) - 1$, we construct a solution on the interval $[\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1})$ by considering the following IVP:

$$v'(w) \;=\; \frac{m'H^{-1}m}{2\sigma^2}\, v^2(w) \;-\; \frac{2\mu}{\sigma^2}\, v(w) \;-\; \frac{2}{\sigma^2}\, \phi^-_{k^\star(l_\gamma)-k}(w) \;+\; \frac{2\gamma}{\sigma^2}, \quad w \in [\tau^-_{k^\star(l_\gamma)-k}, \infty), \tag{124}$$

$$v(\tau^-_{k^\star(l_\gamma)-k}) \;=\; \hat v^-_{\gamma,\,k^\star(l_\gamma)-k+1}(\tau^-_{k^\star(l_\gamma)-k}) \tag{125}$$

where

$$\phi^-_{k^\star(l_\gamma)-k}(w) \;\coloneqq\; \frac{\hat\beta_{i^\star(k^\star(l_\gamma)-k)}}{m_{i^\star(k^\star(l_\gamma)-k)}} \left(\tau^-_{k^\star(l_\gamma)-k-1} - w\right) + \sum_{j=1}^{k^\star(l_\gamma)-k-1} \hat\beta_{i^\star(j)} \lambda^\star_{i^\star(j)} \delta_{i^\star(j)}, \quad w \in [\tau^-_{k^\star(l_\gamma)-k}, \infty), \tag{126}$$

is the linear function that coincides with $\phi(w,y)$ for $w \in [\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}]$ and $y \in (-\infty, \kappa]$. As in Step 1, we apply the results from Section B.1 to obtain a solution. To be specific, consider the Riccati equation (99)–(100) with the following constants:

$$
\begin{aligned}
c_0 &= -\frac{2}{\sigma^2}\Big(\frac{\hat{\beta}_{i^\star(k^\star(l_\gamma)-k)}}{m_{i^\star(k^\star(l_\gamma)-k)}}\big(\tau^-_{k^\star(l_\gamma)-k-1} + \hat{v}^-_{\gamma,\,k^\star(l_\gamma)-k+1}(\tau^-_{k^\star(l_\gamma)-k-1})\big) + \sum_{j=1}^{k^\star(l_\gamma)-k-1}\hat{\beta}_{i^\star(j)}\lambda^\star_{i^\star(j)}\delta_{i^\star(j)} - \gamma\Big),\\
c_1 &= \frac{2}{\sigma^2}\frac{\hat{\beta}_{i^\star(k^\star(l_\gamma)-k)}}{m_{i^\star(k^\star(l_\gamma)-k)}}, \quad c_2 = -\frac{2\mu}{\sigma^2}, \quad c_3 = 0, \quad c_4 = \frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 = \hat{v}^-_{\gamma,\,k^\star(l_\gamma)-k+1}(\tau^-_{k^\star(l_\gamma)-k}).
\end{aligned}
$$

By Lemma 4, there exists a unique solution $y^-_{\gamma,\,k^\star(l_\gamma)-k} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat{v}^-_{\gamma,\,k^\star(l_\gamma)-k} \in C^1[\tau^-_{k^\star(l_\gamma)-k},\infty)$ given by

$$
\hat{v}^-_{\gamma,\,k^\star(l_\gamma)-k}(w) \coloneqq y^-_{\gamma,\,k^\star(l_\gamma)-k}(w - \tau^-_{k^\star(l_\gamma)-k}), \quad w \in [\tau^-_{k^\star(l_\gamma)-k},\infty) \tag{127}
$$

is the unique solution to (124)–(125).

Step 3: Solution on $[w_0, w_1]$**.** Given the solution $\hat{v}^-_{\gamma,1}$ on the interval $[\tau_1, w_0)$, we construct a solution on the interval $[w_0, w_1]$ by considering the following IVP:

$$
v'(w) = \frac{m'H^{-1}m}{2\sigma^2}\, v^2(w) - \frac{2\mu}{\sigma^2}\, v(w) + \frac{2\gamma}{\sigma^2}, \quad w \in [w_0,\infty), \tag{128}
$$

$$
v(w_0) = \hat{v}^-_{\gamma,1}(w_0). \tag{129}
$$

Similar to Steps 1–2, we apply the results from Section B.1 to obtain a solution. To be specific, consider the Riccati equation (99)–(100) with the following constants:

$$
c_0 = \frac{2\gamma}{\sigma^2}, \quad c_1 = 0, \quad c_2 = -\frac{2\mu}{\sigma^2}, \quad c_3 = 0, \quad c_4 = \frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 = \hat{v}^-_{\gamma,1}(w_0).
$$

By Lemma 4, there exists a unique solution $y^-_{\gamma,0} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat{v}^-_{\gamma,\,0} \in C^1[w_0,\infty)$ given by

$$
\hat{v}^-_{\gamma,\,0}(w) \coloneqq y^-_{\gamma,0}(w - w_0), \quad w \in [w_0,\infty) \tag{130}
$$

is the unique solution to (128)–(129).

Step 4: Solution on $[l_\gamma, w_1]$**.** For each $\gamma \geq 0$, we define the function $v^-_\gamma : [l_\gamma, w_1] \to \mathbb{R}$ by pasting together the solutions from Steps 1–3 as follows:

$$
v^-_\gamma(w) \coloneqq \begin{cases} \hat{v}^-_{\gamma,\,k^\star(l_\gamma)}(w), & w \in [l_\gamma, \tau^-_{k^\star(l_\gamma)-1}),\\ \hat{v}^-_{\gamma,\,k^\star(l_\gamma)-k}(w), & w \in [\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}), \quad k = 1,\ldots,k^\star(l_\gamma)-1,\\ \hat{v}^-_{\gamma,0}(w), & w \in [w_0, w_1], \end{cases} \tag{131}
$$

where the functions $\hat{v}^-_{\gamma,\,k^\star(l_\gamma)-k}$ are given by (123), (127), and (130). Next, we show that v^-_γ solves (116)–(117) for appropriate values of γ. As a preliminary, note that by Corollary 4, the effective state cost function

$\phi(w, y)$ for $w \in [l_\gamma, w_1]$ and $y \in (-\infty, \kappa]$ can be rewritten as follows:

$$\phi(w,y) \;=\; \begin{cases} \phi^-_{k^\star(l_\gamma)}(w), & w \in [l_\gamma, \tau^-_{k^\star(l_\gamma)-1}), \\ \phi^-_{k^\star(l_\gamma)-k}(w), & w \in [\tau^-_{k^\star(l_\gamma)-k}, \tau^-_{k^\star(l_\gamma)-k-1}), \quad k \;=\; 1, \ldots, k^\star(l_\gamma)-1, \\ 0, & w \in [w_0, w_1], \end{cases} \tag{132}$$

where the functions ϕ^-_k are given by (122) and (126). We now consider the following IVP:

$$v'(w) \;=\; \frac{m' H^{-1} m}{2\sigma^2}\, v^2(w) \;-\; \frac{2\mu}{\sigma^2}\, v(w) \;-\; \frac{2}{\sigma^2}\, \phi(w,0) \;+\; \frac{2\gamma}{\sigma^2}, \quad w \in [l_\gamma, w_1], \tag{133}$$

$$v(l_\gamma) \;=\; 0. \tag{134}$$

Lemma 10. *For each $\gamma \geq 0$, the function $v^-_\gamma : [l_\gamma, w_1] \to \mathbb{R}$ defined by (131) is continuously differentiable and is the unique solution to (133)–(134).*

Proof. To show that v^-_γ is continuously differentiable on $[l_\gamma, w_1]$, we must establish both continuity and existence of a continuous derivative. First, continuity of v^-_γ on the interior of each subinterval in (119) follows from the continuity of the functions $\hat{v}^-_{\gamma,\,k}$ for $k = 0, 1, \ldots, k^\star(l_\gamma)$ on their respective domains. Continuity of v^-_γ at the endpoints of these subintervals follows directly from the initial conditions (121), (125), and (129). The existence of a continuous derivative follows from the continuity of the functions $\hat{v}^-_\gamma$ and $\phi(\cdot, 0)$, together with (120)–(121), (124)–(125), and (128)–(129). This establishes that $v^-_\gamma \in C^1[l_\gamma, w_1]$. Finally, it follows from the construction in (120)–(132) that v^-_γ satisfies (133)–(134). The uniqueness of v^-_γ follows from the uniqueness of each function $\hat{v}^-_{\gamma,\,k}$ on its respective domain. □

Next, we establish several structural properties of v^-_γ that will be essential in proving that it satisfies the IVP (116)–(117), as well as the existence and uniqueness of a solution to the Bellman equation (60)–(61)..

Lemma 11. *For each $\gamma > 0$, the function $v^-_\gamma \in C^1[l_\gamma, w_1]$ satisfies $(v^-_\gamma)'(l_\gamma) = 0$ and $(v^-_\gamma)'(w) > 0$ for all $w \in (l_\gamma, w_1]$. Consequently, v^-_γ is strictly increasing for each $\gamma > 0$.*

Proof. Fix $\gamma > 0$. Since $v^-_\gamma(l_\gamma) = 0 \leq \kappa$ by (121), it follows from (118) and (132) that $\phi^-_{k^\star(l_\gamma)}(l_\gamma) = \phi(l_\gamma, 0) = \gamma$. Substituting this into (120)–(121) yields $(v^-_\gamma)'(l_\gamma) = 0$. To complete the proof, we must show that $(v^-_\gamma)'(w) > 0$ for all $w \in (l_\gamma, w_1]$. By Lemma 8 and (132), $(v^-_\gamma)'$ is almost everywhere differentiable on $[l_\gamma, w_1]$ with respect to Lebesgue measure. In particular, the second derivative of v^-_γ is given by

$$(v^-_\gamma)''(w) = \frac{m' H^{-1} m}{\sigma^2}\, v^-_\gamma(w)\, (v^-_\gamma)'(w) - \frac{2\mu}{\sigma^2}\, (v^-_\gamma)'(w) - \frac{2}{\sigma^2}\, \phi'(w,0), \;\; w \in [l_\gamma, w_1] \setminus \bigcup_{k=1}^{k^\star(l_\gamma)} \{\tau^-_{k^\star(l_\gamma)-k}\}, \tag{135}$$

with $(v^-_\gamma)''(l_\gamma) = 2\, \hat{\beta}_{i^\star(k^\star(l_\gamma))} \,/\, (m_{i^\star(k^\star(l_\gamma))}\, \sigma^2) > 0$. From (135), continuity of v^-_γ and $(v^-_\gamma)'$, and continuity of $\phi'(\cdot, 0)$ in a neighborhood of l_γ, we conclude that $(v^-_\gamma)''$ is continuous in a neighborhood of l_γ. Since $(v^-_\gamma)''(l_\gamma) > 0$, there exists $\bar{w} > l_\gamma$ such that $(v^-_\gamma)''(w) > 0$ for all $w \in [l_\gamma, \bar{w}]$. From this and $(v^-_\gamma)'(l_\gamma) = 0$, it follows that $(v_\gamma^-)'(w) > 0$ for all $w \in (l_\gamma, \bar{w}]$. It remains to show that $(v_\gamma^-)'(w) > 0$ for all $w \in (\bar{w}, w_1]$. Aiming for a contradiction, suppose that $(v_\gamma^-)'(w)$ is not strictly positive on $(\bar{w}, w_1]$. It follows that

$$\hat{w} \coloneqq \inf \big\{ w \in (\bar{w}, w_1] : (v_\gamma^-)'(w) = 0 \big\}$$

is well-defined. From the definition of $\hat{w}$, we have that $(v_\gamma^-)'(\hat{w}) = 0$ and $(v_\gamma^-)'(w) > 0$ for all $w \in (l_\gamma, \hat{w})$. Since $v_\gamma^-(l_\gamma) = 0$ by (121), we must have $v_\gamma^-(\hat{w}) > 0$. Substituting this into (133) yields

$$\frac{\sigma^2}{2} (v_\gamma^-)'(\hat{w}) \, = \, \gamma \, - \, \phi(\hat{w}, 0) \, - \, \mu \, v_\gamma^-(\hat{w}) \, + \, \frac{m' H^{-1} m'}{4} (v_\gamma^-)^2(\hat{w}) \, > \, 0,$$

where the inequality follows from $\gamma = \phi(l_\gamma, 0) > \phi(\hat{w}, 0)$ (which follows from Lemma 8 and $l_\gamma < \bar{w} \wedge w_0 \leq \hat{w} \leq w_1$) and the fact that the third and fourth terms in the equality are strictly positive (since $\mu < 0$ and $m' H^{-1} m' > 0$). This contradicts $(v_\gamma^-)'(\hat{w}) = 0$. Hence, the assumption that $(v_\gamma^-)'(w)$ is not strictly positive on $(\bar{w}, w_1]$ is incorrect, which completes the proof. □

Lemma 12. *The mapping $\gamma \mapsto v_\gamma^-$ is strictly increasing in the following sense: For $0 \leq \gamma_1 < \gamma_2$, we have that $v_{\gamma_2}^-(w) > v_{\gamma_1}^-(w)$ for all $w \in [l_{\gamma_1}, w_1]$.*

Proof. Fix γ_1 and γ_2 such that $0 \leq \gamma_1 < \gamma_2$. Since $l_{\gamma_2} < l_{\gamma_1}$ and $v_{\gamma_2}^-$ is strictly increasing (by Lemma 11) with $v_{\gamma_2}^-(l_{\gamma_2}) = 0$ (by (121)), it follows that $v_{\gamma_2}^-(l_{\gamma_1}) > 0 = v_{\gamma_1}^-(l_{\gamma_1})$. Aiming for a contradiction, suppose that there exists some $\bar{w} \in (l_{\gamma_1}, w_1]$ such that $v_{\gamma_1}^-(\bar{w}) \geq v_{\gamma_2}^-(\bar{w})$. It follows that

$$\hat{w} \coloneqq \inf \big\{ l_{\gamma_1} < w \leq w_1 \, : \, v_{\gamma_1}^-(w) \geq v_{\gamma_2}^-(w) \big\}$$

is well-defined. By the intermediate value theorem, the continuity of $v_{\gamma_1}^-$ and $v_{\gamma_2}^-$, and the definition of $\hat{w}$, we have $v_{\gamma_1}^-(\hat{w}) = v_{\gamma_2}^-(\hat{w})$ and $v_{\gamma_2}^-(w) > v_{\gamma_1}^-(w)$ for all $w \in [l_{\gamma_1}, \hat{w})$. From (133), it follows that

$$(v_{\gamma_2}^-)'(w) - (v_{\gamma_1}^-)'(w) \, > \, \frac{2}{\sigma^2} (\gamma_2 - \gamma_1) \, > \, 0 \quad \text{for all} \quad w \in [l_{\gamma_1}, \hat{w}).$$

However, this and $v_{\gamma_2}^-(l_{\gamma_1}) > v_{\gamma_1}^-(l_{\gamma_1})$ imply that $v_{\gamma_2}^-(\hat{w}) > v_{\gamma_1}^-(\hat{w})$, which is a contradiction. □

Lemma 13. *The mapping $\gamma \mapsto v_\gamma^-(w_1)$ is strictly increasing with $v_0^-(w_1) = 0$ and $\lim_{\gamma \to \infty} v_\gamma^-(w_1) = \infty$.*

Proof. By Lemma 12, the mapping $\gamma \mapsto v_\gamma^-(w_1)$ is strictly increasing. To show that $v_0^-(w_1) = 0$, note that $l_0 = w_0$, so it follows by (131) that $v_0^- \in C^1[w_0, w_1]$ is the unique solution to the IVP from Step 3 of the construction, i.e.,

$$\begin{aligned} v'(w) \, &= \, \frac{m' H^{-1} m}{2\sigma^2} v^2(w) \, - \, \frac{2\mu}{\sigma^2} v(w), \quad w \in [w_0, \infty), \\ v(w_0) \, &= \, 0. \end{aligned}$$

Since the zero function satisfies this IVP, uniqueness implies that that $v_0^-(w) = 0$ for all $w \in [w_0, w_1]$, and in particular, $v_0^-(w_1) = 0$, as desired. Finally, to prove that $\lim_{\gamma \to \infty} v_\gamma^-(w_1) = \infty$, fix an arbitrary $M > 0$.

Then, choosing $\gamma = M\sigma^2/(2(w_1 - w_0)) > 0$, it follows that

$$
\begin{aligned}
v_\gamma^-(w_1) &= \int_{l_\gamma}^{w_1} (v_\gamma^-)'(w)\,dw \geq \int_{w_0}^{w_1} (v_\gamma^-)'(w)\,dw = \int_{w_0}^{w_1} (\hat{v}_{\gamma,0}^-)'(w)\,dw \\
&= \int_{w_0}^{w_1} \Big(\frac{m'H^{-1}m}{2\sigma^2}\,(\hat{v}_{\gamma,0}^-)^2(w) \;-\; \frac{2\mu}{\sigma^2}\,\hat{v}_{\gamma,0}^-(w) \;+\; \frac{2\gamma}{\sigma^2}\Big)\,dw \;\geq\; \frac{2\gamma}{\sigma^2}\,(w_1 - w_0) \;=\; M,
\end{aligned}
$$

where the first equality follows from $v_\gamma^-(l_\gamma) = 0$ by (121), the first inequality follows from the fact that $(v_\gamma^-)'(w) > 0$ for all $w \in [l_\gamma, w_1]$ by Lemma 11, the second equality follows from (131), the third equality follows from (128), the second inequality follows from the fact that the first two terms in the third equality are nonnegative, and the final equality follows from the choice of γ. Since M was arbitrary, we conclude that $\lim_{n\to\infty} v_\gamma^-(w_1) = \infty$. ☐

Lemma 14. *The mapping $\gamma \mapsto v_\gamma^-(w_1)$ is continuous on $[0, \infty)$.*

Proof. We show that for all $\gamma \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $\gamma' \in (\gamma - \delta, \gamma + \delta) \cap [0, \infty)$, we have $|v_\gamma^-(x) - v_{\gamma'}^-(x)| < \epsilon$ for all $x \in [w_0, w_1]$. The desired result then follows by setting $x = w_1$. Fix $\gamma \geq 0$ and $\epsilon > 0$, and consider some $\delta \in (0, 1)$ to be specified later. For any $\gamma' \in (\gamma - \delta, \gamma + \delta) \cap [0, \infty)$ and $x \in [w_0, w_1]$, we have that

$$
\begin{aligned}
v_\gamma^-(x) \;-\; v_{\gamma'}^-(x) &= \int_{l_\gamma}^{x} (v_\gamma^-)'(w)\,dw - \int_{l_\gamma}^{x} (v_{\gamma'}^-)'(w)\,dw \\
&= \int_{l_\gamma}^{x} \Big[\frac{m'H^{-1}m}{2\sigma^2}\,(v_\gamma^-)^2(w) - \frac{2\mu}{\sigma^2}\,v_\gamma^-(w)\Big]\,dw - \int_{l_{\gamma'}}^{x} \Big[\frac{m'H^{-1}m}{2\sigma^2}\,(v_{\gamma'}^-)^2(w) - \frac{2\mu}{\sigma^2}\,v_{\gamma'}^-(w)\Big]\,dw \\
&\quad + \int_{l_\gamma}^{x} \frac{2}{\sigma^2}\,\phi(w,0)\,dw - \int_{l_{\gamma'}}^{x} \frac{2}{\sigma^2}\,\phi(w,0)\,dw \;+\; (x - l_\gamma)\,\frac{2\gamma}{\sigma^2} \;-\; (x - l_{\gamma'})\,\frac{2\gamma'}{\sigma^2},
\end{aligned}
$$

where the first equality follows from the fundamental theorem of calculus and (121), and the second equality from (133) and rearranging terms. Taking absolute values of both sides of the previous display then gives

$$
\begin{aligned}
|v_\gamma^-(x) \;-\; v_{\gamma'}^-(x)| &\leq \int_{l_\gamma \wedge l_{\gamma'}}^{l_\gamma \vee l_{\gamma'}} \Big|\, \frac{m'H^{-1}m}{2\sigma^2}\,(v_{\gamma\vee\gamma'}^-)^2(w) \;-\; \frac{2\mu}{\sigma^2}\,v_{\gamma\vee\gamma'}^-(w) \;+\; \frac{2}{\sigma^2}\,\phi(w,0)\Big|\,dw \\
&\quad + \int_{l_\gamma \vee l_{\gamma'}}^{x} \frac{m'H^{-1}m}{2\sigma^2}\,\big|(v_\gamma^-)^2(w) - (v_{\gamma'}^-)^2(w)\big|\,dw \;+\; \Big|(x - l_\gamma)\frac{2\gamma}{\sigma^2} - (x - l_{\gamma'})\frac{2\gamma'}{\sigma^2}\Big| \\
&\leq \underbrace{\Big|\frac{m'H^{-1}m}{2\sigma^2}\,(v_{\gamma+1}^-)^2(w_1) - \frac{2\mu}{\sigma^2}\,v_{\gamma+1}^-(w_1) + \frac{2}{\sigma^2}\,\phi(l_{\gamma+1},0)\Big|}_{=:\,C_1(\gamma)} \cdot |l_\gamma - l_{\gamma'}| \\
&\quad + \underbrace{\frac{2}{\sigma^2}\,|w_1 - l_{\gamma+1}|}_{=:\,C_2(\gamma)} \cdot |\gamma - \gamma'| + \int_{l_\gamma \vee l_{\gamma'}}^{x} \underbrace{\Big(\frac{m'H^{-1}m}{\sigma^2}\,v_{\gamma+1}^-(w_1) - \frac{2\mu}{\sigma^2}\Big)}_{=:\,C_3(\gamma)} |v_\gamma^-(w) - v_{\gamma'}(w)|\,dw \\
&= \; C_1(\gamma)\,|l_\gamma - l_{\gamma'}| \;+\; C_2(\gamma)\,|\gamma - \gamma'| \;+\; C_3(\gamma) \int_{l_\gamma \vee l_{\gamma'}}^{x} |v_\gamma^-(w) - v_{\gamma'}(w)|\,dw,
\end{aligned}
$$

where the first inequality follows from the triangle inequality and rearranging terms, and the second inequality from Lemmas 8, 11, and 12 and the inequality $l_{\gamma+1} \le l_\gamma \wedge l_{\gamma'}$.[19] Furthermore, by Lemma 8, the mapping $w \mapsto \phi(w, 0)$ is continuous and injective, implying that $l_{\gamma'} \to l_\gamma$ as $\gamma' \to \gamma$.[20] Therefore, there exists some $\delta \in (0, 1)$ such that for all $\gamma' \in (\gamma - \delta, \gamma + \delta) \cap [0, \infty)$ we have

$$\widetilde{C}(\gamma, \gamma') \coloneqq C_1(\gamma)\,|l_\gamma - l_{\gamma'}| \,+\, C_2(\gamma)\,|\gamma - \gamma'| \,<\, \epsilon \exp\big\{-C_3(\gamma)\,(w - l_{\gamma+1})\big\}.$$

Then, following the same argument as in the proof of Gronwall's inequality (see, e.g., Theorem 1.9.1 in Lakshmikantham and Leela (1969)), it follows from the previous two displayed equations that

$$|v_\gamma^-(x) \,-\, v_{\gamma'}(x)| \,\le\, \widetilde{C}(\gamma, \gamma') \exp\Big(\int_{l_\gamma \vee l_{\gamma'}}^{x} C_3(\gamma)\, dw\Big) \,\le\, \widetilde{C}(\gamma, \gamma') \exp\big\{C_3(\gamma)\,(w_1 - l_{\gamma+1})\big\} \,<\, \epsilon.$$

for all $x \in [w_0, w_1]$. This completes the proof. □

Equipped with the structural properties of v_γ^- established above, we now show that there exists a solution to the IVP (116)–(117). To that end, Lemma 13 ensures that the following is well-defined:

$$\gamma_\kappa \coloneqq \sup\big\{\gamma \ge 0 \,:\, v_\gamma^-(w_1) \le \kappa\big\}. \tag{136}$$

The next result shows that γ_κ serves as an upper bound on the values of γ for which we can prove the existence of a solution to the IVP (116)–(117). However, as we will see in the subsequent sections, this restriction on γ does not prohibit us from finding the unique solution to the Bellman equation (60)–(61).

Lemma 15. *For $\gamma \in [0, \gamma_\kappa]$, the function $v_\gamma^- \in C^1[l_\gamma, w_1]$ is the unique solution to (116)–(117).*

Proof. For $\gamma \in [0, \gamma_\kappa]$, it follows from Lemma 12 and (136) that $v_\gamma^-(w) \le \kappa$ for all $w \in [l_\gamma, w_1]$. Thus, $\phi(w, v_\gamma^-(w)) = \phi(w, 0)$ for all $w \in [l_\gamma, w_1]$. From this and Lemma 10, it follows that v_γ^- is the unique solution to (116)–(117), completing the proof. □

B.2.3.2 IVP on $[w_1, \infty)$.

For each $\gamma \ge 0$, consider the following IVP:

$$v'(w) \,=\, \frac{m' H^{-1} m}{2\sigma^2}\, v^2(w) \,-\, \frac{2\mu}{\sigma^2}\, v(w) \,-\, \frac{2}{\sigma^2}\, \phi(w, v(w)) \,+\, \frac{2\gamma}{\sigma^2}, \quad w \in [w_1, u_\gamma], \tag{137}$$

$$v(u_\gamma) \,=\, \kappa, \tag{138}$$

[19] In particular, for the second inequality, since $0 < \delta < 1$, Lemmas 11 and 12 imply that $v_{\gamma+1}(w_1) \ge \max\big\{\sup\{v_\gamma^-(w) : w \in [l_\gamma, w_1]\}, \sup\{v_{\gamma'}^-(w) : w \in [l_{\gamma'}, w_1]\}\big\}$. Furthermore, by Lemma 8, $\phi(l_{\gamma+1}, 0) \ge \phi(w, 0)$ for all $w \in [l_\gamma \wedge l_{\gamma'}, w_1]$ since $l_{\gamma+1} \le l_\gamma \wedge l_{\gamma'}$.

[20] To be more precise, the function $\phi(\cdot, 0)$ restricted to $(-\infty, w_0]$ is continuous and injective. Therefore, it has a continuous inverse $(\phi(\cdot, 0)|_{(-\infty, w_0]})^{-1} : [0, \infty) \to (-\infty, w_0]$. Denoting this inverse as ϕ^{-1}, it follows that for any sequence $\{\gamma_n\} \subseteq [0, \infty)$ such that $\gamma_n \to \gamma$ as $n \to \infty$, we have $l_{\gamma_n} = \phi^{-1}(\gamma_n) \to \phi^{-1}(\gamma) = l_\gamma$ as $n \to \infty$, proving that $\gamma \mapsto l_\gamma$ is continuous, as desired.

where the upper barrier $u_\gamma \in \mathbb{R}$ is given by[21]

$$u_\gamma \;:=\; \inf\Big\{w \geq w_1 \;:\; \phi(w,\kappa) \;=\; \gamma \;+\; \frac{m'H^{-1}m}{4}\kappa^2 \;-\; \mu\kappa\Big\}. \tag{139}$$

It follows from (139) that any solution $v \in C^1[w_1, u_\gamma]$ to (137)–(138) satisfies $v'(u_\gamma) = 0$. Therefore, we interpret the solution to this IVP as the value function associated with a barrier policy (with an upper barrier at u_γ) that achieves a long-run average cost of γ and satisfies the smooth pasting condition at the upper barrier; see Proposition 3.

We next prove the existence of a closed-form solution to the IVP (137)–(138) for all $\gamma \geq 0$. In particular, the solution is constructed iteratively over subintervals where the product class $k^\star(w,\cdot)$ for $w \in [w_1,\infty)$ remains constant, allowing us to leverage the results from Section B.1 to derive a closed-form solution. We proceed in four main steps. In Step 1, we find the unique solution on $(-\infty, u_\gamma]$, starting from the boundary condition at u_γ, and determine the first point $\tau^+_{\gamma,1}$ where the solution reaches y_1, i.e., the first point at which $k^\star(w,\cdot)$ changes; see (115). In Step 2, we use y_1 as the initial value and find the unique solution on $(-\infty, \tau^+_{\gamma,1}]$, and determine the point $\tau^+_{\gamma,2}$ where the solution reaches the y_2, i.e., the second point at which $k^\star(w,\cdot)$ changes. This process continues iteratively, where the y_j value associated with the previous interval provides the initial value for the next, until we obtain solutions on $(-\infty, \tau^+_{\gamma,j}]$ for $j = 1,\dots,b-1$, where $b \in \mathbb{N}$ is the number of breakpoints of $\mathbf{A}$; see Section B.2.2. In Step 3, we use y_b as the initial value and find the unique solution on $(-\infty, \tau^+_{\gamma,b}]$. In Step 4, we combine the solutions from Steps 1–3 to obtain a continuously differentiable function on $[w_1, u_\gamma]$. After constructing this function, we establish its key properties and show that it uniquely solves the IVP (137)–(138).

Step 1: Solution on $(-\infty, u_\gamma]$. We construct a solution on the interval $(-\infty, u_\gamma]$ by considering the following IVP:

$$v'(w) \;=\; \frac{m'H^{-1}m}{2\sigma^2}\,v^2(w) \;-\; \frac{2\mu}{\sigma^2}\,v(w) \;-\; \frac{2}{\sigma^2}\,\phi_0^+(w, v(w)) \;+\; \frac{2\gamma}{\sigma^2}, \quad w \in (-\infty, u_\gamma], \tag{140}$$

$$v(u_\gamma) \;=\; \kappa, \tag{141}$$

where

$$\phi_0^+(w,y) \;:=\; \big[\alpha_{k_0} + \ell_{k_0}\big(d_{k_0} - m_{k_0}y\big)\big](w - w_1)/m_{k_0}, \quad (w,y) \in (-\infty, u_\gamma] \times \mathbb{R}. \tag{142}$$

is the function that coincides with $\phi(w,y)$ for $w \in [w_1,\infty)$ and $y \in (y_1,\kappa]$. We next apply the results from Section B.1 to obtain a solution to (140)–(141). To be specific, consider the Riccati equation (99)–(100) with

[21]Since $\gamma + \frac{m'H^{-1}m}{4}\kappa^2 - \mu\kappa > 0$ for all $\gamma \geq 0$, the existence of u_γ follows from Lemma 8 and the fact that the mapping $w \mapsto \phi(w,\kappa)$ is continuous and strictly increasing on $[w_1,\infty)$, with $\phi(w_1,\kappa) = 0$ and $\phi(w,\kappa) \to \infty$ as $w \to \infty$. This also ensures that $u_\gamma > w_1$ for all $\gamma \geq 0$.

the following constants:

$$c_0 = -\frac{2}{\sigma^2}\Big(\frac{\alpha_{k_0} + \ell_{k_0} d_{k_0}}{m_{k_0}} + \gamma\Big), \quad c_1 = -\frac{2\,(\alpha_{k_0} + \ell_{k_0} d_{k_0})}{\sigma^2 m_{k_0}},$$
$$c_2 = \frac{2}{\sigma^2}\big(\mu + \ell_{k_0}(w_1 + u_\gamma)\big), \quad c_3 = \frac{2\ell_{k_0}}{\sigma^2}, \quad c_4 = -\frac{m' H^{-1} m}{2\sigma^2}, \quad \text{and} \quad y_0 = \kappa.$$

By Lemma 5, there exists a unique solution $y^+_{\gamma,0} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat{v}^+_{\gamma,0} \in C^1(-\infty, u_\gamma]$ given by

$$\hat{v}^+_{\gamma,0}(w) \coloneqq y^+_{\gamma,0}(u_\gamma - w), \quad w \in (-\infty, u_\gamma] \tag{143}$$

is the unique solution to (140)–(141). To construct the solution on the next interval, define $\tau^+_{\gamma,1} \coloneqq \sup\{w \leq u_\gamma : \hat{v}^+_{\gamma,0}(w) \leq y_1\}$, and note that by continuity of $\hat{v}^+_{\gamma,0}$, we have $\hat{v}^+_{\gamma,0}(\tau^+_{\gamma,1}) = y_1$. For notational convenience in the next step, define $\tau^+_{\gamma,0} \coloneqq u_\gamma$.

Step 2: Solution on $(-\infty, \tau^+_{\gamma,j}]$ for $j = 1, \ldots, b-1$. Given the solution $\hat{v}^+_{\gamma,j-1}$ on the interval $(-\infty, \tau^+_{\gamma,j-1}]$ for $j = 1, \ldots, b-1$, we construct a solution on the interval $(-\infty, \tau^+_{\gamma,j}]$ by considering the following IVP:

$$v'(w) = \frac{m' H^{-1} m}{2\sigma^2}\, v^2(w) - \frac{2\mu}{\sigma^2}\, v(w) - \frac{2}{\sigma^2}\, \phi^+_j(w, v(w)) + \frac{2\gamma}{\sigma^2}, \quad w \in (-\infty, \tau^+_{\gamma,j}], \tag{144}$$
$$v(\tau^+_{\gamma,j}) = y_j, \tag{145}$$

where

$$\phi^+_j(w,y) \coloneqq \big[\alpha_{k_j} + \ell_{k_j}\big(d_{k_j} - m_{k_j} y\big)\big](w - w_1)/m_{k_j}, \quad (w,y) \in (-\infty, \tau^+_{\gamma,j}] \times \mathbb{R}. \tag{146}$$

is the function that coincides with $\phi(w,y)$ for $w \in [w_1, \infty)$ and $y \in (y_{j+1}, y_j]$. As in Step 1, we apply the results from Section B.1 to obtain a solution. To be specific, consider the Riccati equation (99)–(100) with the following constants:

$$c_0 = -\frac{2}{\sigma^2}\Big(\frac{\alpha_{k_j} + \ell_{k_j} d_{k_j}}{m_{k_j}} + \gamma\Big), \quad c_1 = -\frac{2\,(\alpha_{k_j} + \ell_{k_j} d_{k_j})}{\sigma^2 m_{k_j}},$$
$$c_2 = \frac{2\mu}{\sigma^2}\big(\mu + \ell_{k_j}(w_1 + \tau^+_{\gamma,j})\big), \quad c_3 = \frac{2\ell_{k_j}}{\sigma^2}, \quad c_4 = -\frac{m' H^{-1} m}{2\sigma^2}, \quad \text{and} \quad y_0 = y_j.$$

By Lemma 5, there exists a unique solution $y^+_{\gamma,j} \in C^1[0,\infty)$ to this Riccati equation. It then follows that the function $\hat{v}^+_{\gamma,j} \in C^1(-\infty, \tau^+_{\gamma,j}]$ given by

$$\hat{v}^+_{\gamma,j}(w) \coloneqq y^+_{\gamma,j}(\tau^+_{\gamma,j} - w), \quad w \in (-\infty, \tau^+_{\gamma,j}] \tag{147}$$

is the unique solution to (144)–(145). To construct the solution on the next interval, define $\tau^+_{\gamma,j+1} \coloneqq \sup\{w \leq \tau^+_{\gamma,j} : \hat{v}^+_{\gamma,j}(w) \leq y_{j+1}\}$, and note that by continuity of $\hat{v}^+_{\gamma,j}$, we have $\hat{v}^+_{\gamma,j}(\tau^+_{\gamma,j+1}) = y_{j+1}$.

Step 3: Solution on $(-\infty, \tau^+_{\gamma,b}]$. Given the solution $\hat{v}^+_{\gamma,b-1}$ on the interval $(-\infty, \tau^+_{\gamma,b-1}]$, we construct a solution on the interval $(-\infty, \tau^+_{\gamma,b}]$ by considering the following IVP:

$$v'(w) \;=\; \frac{m'H^{-1}m}{2\sigma^2}\, v^2(w) \;-\; \frac{2\mu}{\sigma^2}\, v(w) \;-\; \frac{2}{\sigma^2}\, \phi^+_b(w, v(w)) \;+\; \frac{2\gamma}{\sigma^2}, \quad w \in (-\infty, \tau^+_{\gamma,b}], \tag{148}$$

$$v(\tau^+_{\gamma,b}) \;=\; y_b, \tag{149}$$

where

$$\phi^+_b(w, y) \;\coloneqq\; \alpha_{k_b}(w - w_1)/m_{k_b}, \quad (w, y) \in (-\infty, \tau^+_{\gamma,b}] \times \mathbb{R} \tag{150}$$

is the function that coincides with $\phi(w, y)$ for $w \in [w_1, \infty)$ and $y \in (-\infty, y_b]$. As in Steps 1–2, we apply the results from in Section B.1 to obtain a solution. To be specific, consider the Riccati equation (99)–(100) with the following constants:

$$c_0 \;=\; -\frac{2}{\sigma^2}\Big(\gamma + w_1 - \frac{\alpha_{k_b}\tau^+_{\gamma,k_b}}{m_{k_b}}\Big), \quad c_1 \;=\; -\frac{2\alpha_{k_b}}{\sigma^2 m_{k_b}},$$
$$c_2 \;=\; \frac{2\mu}{\sigma^2}, \quad c_3 \;=\; 0, \quad c_4 \;=\; -\frac{m'H^{-1}m}{2\sigma^2}, \quad \text{and} \quad y_0 \;=\; y_b.$$

By Lemma 4, there exists a unique solution $y^+_{\gamma,b} \in C^1[0, \infty)$ to this Riccati equation. It then follows that the function $\hat{v}^+_{\gamma,b} \in C^1(-\infty, \tau^+_{\gamma,b}]$ given by

$$\hat{v}^+_{\gamma,b}(w) \;\coloneqq\; y^+_{\gamma,b}(\tau^+_{\gamma,b} - w), \quad w \in (-\infty, \tau^+_{\gamma,b}] \tag{151}$$

is the unique solution to (148)–(149).

Step 4: Solution on $[w_1, u_\gamma]$. For each $\gamma \geq 0$, we define the function $v^+_\gamma : [w_1, u_\gamma] \to \mathbb{R}$ by pasting together the solutions from Steps 1–3 as follows:

$$v^+_\gamma(w) \;\coloneqq\; \begin{cases} \hat{v}^+_{\gamma,j}(w), & w \in \big(\tau^+_{\gamma,j+1}, \tau^+_{\gamma,j}\big], \quad j \;=\; 0,\, 1,\, \ldots,\, B_\gamma - 1, \\ \hat{v}^+_{\gamma,B_\gamma}(w), & w \in \big[w_1, \tau^+_{\gamma,B_\gamma}\big], \end{cases} \tag{152}$$

where $B_\gamma \coloneqq \max\{j = 1, \ldots, b \,:\, \tau^+_{\gamma,j} > w_1\}$ and the functions $\hat{v}^+_{\gamma,j}$ for $j = 0,\, 1,\, \ldots,\, B_\gamma$ are given by (143), (147), and (151).

The next two results show that v^+_γ is continuously differentiable and strictly increasing, which together ensure that v^+_γ solves the IVP (137)–(138).

Lemma 16. *For each $\gamma \geq 0$, the function $v^+_\gamma : [w_1, u_\gamma] \to \mathbb{R}$ defined by (152) is continuously differentiable.*

Proof. To show that v^+_γ is continuously differentiable on $[w_1, u_\gamma]$, we must prove that it has a continuous derivative. For convenience, we decompose the interval $[w_1, u_\gamma]$ into two disjoint sets, $S_{\gamma,1}$ and $S_{\gamma,2}$, as follows:

$$S_{\gamma,1} \;\coloneqq\; [w_1, \tau^+_{\gamma,\,B_\gamma}) \,\cup\, \bigcup_{j=1}^{B_\gamma - 1} (\tau^+_{\gamma,j+1}, \tau^+_{\gamma,j}) \,\cup\, (\tau^+_{\gamma,1}, u_\gamma] \quad \text{and} \quad S_{\gamma,2} \;\coloneqq\; \bigcup_{j=1}^{B_\gamma} \{\tau^+_{\gamma,j}\}.$$

We first prove that v^+_γ is differentiable. Differentiability on $S_{\gamma,1}$ follows from the differentiability of the functions $\hat{v}^+_{\gamma,\,j}$ for $j = 0, 1, \ldots, B_\gamma$. Differentiability on $S_{\gamma,2}$ follows from

$$\hat{v}^+_{\gamma,j}(\tau^+_{\gamma,j+1}) \;=\; \hat{v}^+_{\gamma,j+1}(\tau^+_{\gamma,j+1}) \quad \text{and} \quad \phi^+_j(\tau^+_{\gamma,j+1}, y_{j+1}) \;=\; \phi^+_{j+1}(\tau^+_{\gamma,j+1}, y_{j+1}), \quad j \;=\; 0,\, 1,\, \ldots,\, B_\gamma - 1,$$

where the first equality follows from (145) and (149), and the second follows from (115), (142), (146), and (150). Next, we prove that the derivative of v^+_γ is continuous. Continuity on $S_{\gamma,1}$ follows from the continuity of the functions $\hat{v}^+_{\gamma,j}$ for $j = 0, 1, \ldots, B_\gamma$ and (140), (142), (144), (146), (148), and (150). Continuity on $S_{\gamma,2}$ follows from the initial conditions (145) and (149). ☐

Lemma 17. *For each $\gamma \geq 0$, the function $v^+_\gamma \in C^1[w_1, u_\gamma]$ satisfies $(v^+_\gamma)'(u_\gamma) = 0$ and $(v^+_\gamma)'(w) > 0$ for all $w \in [w_1, u_\gamma)$. Consequently, v^+_γ is strictly increasing for each $\gamma \geq 0$.*

Proof. Fix $\gamma \geq 0$. Since $\phi(u_\gamma, \kappa) = \phi^+_0(u_\gamma, \kappa)$ by Corollaries 4–5 and (142), it follows from (139)–(141) that $(v^+_\gamma)'(u_\gamma) = 0$. To complete the proof, we must show that $(v^+_\gamma)'(w) > 0$ for all $w \in [w_1, u_\gamma)$.As we will see shortly, it suffices to show that $(\hat{v}^+_{\gamma,0})'(w) > 0$ for all $w \in (-\infty, u_\gamma)$ and that $(\hat{v}^+_{\gamma,j})'(w) > 0$ for all $w \in (-\infty, \tau^+_{\gamma,j}]$ for $j = 1, \ldots, B_\gamma$. To that end, consider the function $\hat{v}^+_{\gamma,0}$. By (140) and (142), it follows that $\hat{v}^+_{\gamma,0}$ has a well-defined second derivative given by

$$(\hat{v}^+_{\gamma,0})''(w) \;=\; \frac{m' H^{-1} m}{\sigma^2}\, \hat{v}^+_{\gamma,0}(w)\, (\hat{v}^+_{\gamma,0})'(w) \;-\; \frac{2\mu}{\sigma^2}\, (\hat{v}^+_{\gamma,0})'(w) \;-\; \frac{2}{\sigma^2} \frac{d}{dw}\, \phi^+_0\big(w, \hat{v}^+_{\gamma,0}(w)\big), \quad w \in (-\infty, u_\gamma],$$

where

$$\frac{d}{dw}\, \phi^+_0\big(w, \hat{v}^+_{\gamma,0}(w)\big) \;=\; \hat{\alpha}_{k_0}\big(\hat{v}^+_{\gamma,0}(w)\big)/m_{k_0} \;+\; (w - w_1)\, \ell_{k_0} (\hat{v}^+_{\gamma,0})'(w), \quad w \in (-\infty, u_\gamma].$$

Since $\hat{v}^+_{\gamma,0}(u_\gamma) = \kappa$ and $(\hat{v}^+_{\gamma,0})'(u_\gamma) = 0$ by (139)–(141), it follows from the above that $(\hat{v}^+_{\gamma,0})''(u_\gamma) = -2\hat{\alpha}_{k_0}(\kappa)/\sigma^2 m_{k_0} < 0$. It follows that $\hat{v}^+_{\gamma,0}$ has a local maximum at u_γ, implying that there exists some $\bar{w}_0 < u_\gamma$ such that $(\hat{v}^+_{\gamma,0}(w))'(w) > 0$ for all $w \in [\bar{w}_0, u_\gamma)$. Now, aiming for a contradiction, suppose that $(\hat{v}^+_{\gamma,0})'$ is not strictly positive on $(-\infty, \bar{w}_0)$. It follows that

$$\hat{w}_0 \;\coloneqq\; \sup\big\{w \in (-\infty, \bar{w}_0) \;:\; (\hat{v}^+_{\gamma,0})'(w) = 0\big\}$$

is well-defined. Therefore, we have that $\hat{v}^+_{\gamma,0}(\hat{w}_0) < \kappa$, and by the continuity of $(\hat{v}^+_{\gamma,0})'$, we also have that $(\hat{v}^+_{\gamma,0})'(\hat{w}_0) = 0$. Similar to before, we have that $(\hat{v}^+_{\gamma,0})''(\hat{w}_0) = -2\,\hat{\alpha}_{k_0}(\hat{v}^+_{\gamma,0}(\hat{w}_0))/\sigma^2 < 0$. It follows that $\hat{v}^+_{\gamma,0}$ has a local maximum at $\hat{w}_0$, contradicting $(\hat{v}^+_{\gamma,0})'(w) > 0$ for all $w \in (\hat{w}_0, u_\gamma)$. Thus, $(\hat{v}^+_{\gamma,0})'(w) > 0$ for all $w \in (-\infty, u_\gamma)$. In particular, by (152) and Lemma 16, it follows that $(v^+_\gamma)'(w) > 0$ for $w \in [\tau^+_{\gamma,1}, u_\gamma)$ and that $(\hat{v}^+_{\gamma,1})'(\tau^+_{\gamma,1}) > 0$. The same argument can then be applied iteratively to show that $(\hat{v}^+_{\gamma,j})'(w) > 0$

for all $w \in (-\infty, \tau^+_{\gamma,j}]$ for $j = 1, \ldots, B_\gamma$. Therefore, by Lemma 16, we conclude that $(v^+_\gamma)'(w) > 0$ for all $w \in [w_1, u_\gamma)$, completing the proof. □

Lemma 18. *For each $\gamma \geq 0$, the function $v^+_\gamma \in C^1[w_1, u_\gamma]$ is the unique solution to (137)–(138).*

Proof. It follows from (141) and (152) that $v^+_\gamma(u_\gamma) = \hat{v}^+_{\gamma,0}(u_\gamma) = \kappa$. Therefore, v^+_γ satisfies (138). It remains to show that v^+_γ satisfies (137). To that end, by Corollary 5 and Lemma 17, we have that

$$k^\star(w, v^+_\gamma(w)) \;=\; \begin{cases} k^+_j, & w \in \big(\tau^+_{\gamma,j+1}, \tau^+_{\gamma,j}\big], \quad j \;=\; 0,\, 1,\, \ldots,\, B_\gamma - 1, \\ k^+_{B_\gamma}, & w \in \big[w_1, \tau^+_{\gamma,B_\gamma}\big]. \end{cases} \tag{153}$$

It then follows from Corollary 4, together with (142), (146), (150), and (153), that

$$\phi(w, v^+_\gamma(w)) \;=\; \begin{cases} \phi^+_j(w, v^+_\gamma(w)), & w \in \big(\tau^+_{\gamma,j+1}, \tau^+_{\gamma,j}\big], \quad j \;=\; 0,\, 1,\, \ldots,\, B_\gamma - 1, \\ \phi^+_{B_\gamma}(w, v^+_\gamma(w)), & w \in \big[w_1, \tau^+_{\gamma,B_\gamma}\big]. \end{cases} \tag{154}$$

Therefore, by (140), (144), (148), and (154), we conclude that v^+_γ satisfies (137). □

Next, we establish several structural properties of v^+_γ that are essential in proving the existence and uniqueness of a solution to the Bellman equation (60)–(61).

Lemma 19. *The mapping $\gamma \mapsto v^+_\gamma(w)$ is strictly decreasing in the following sense: For $0 \leq \gamma_1 < \gamma_2$, we have that $v^+_{\gamma_2}(w) < v^+_{\gamma_1}(w)$ for all $w \in [w_1, u_{\gamma_1}]$.*

Proof. Fix γ_1 and γ_2 such that $0 \leq \gamma_1 < \gamma_2$. Since $u_{\gamma_2} > u_{\gamma_1}$ and $v^+_{\gamma_2}$ is strictly increasing (by Lemma 17) with $v^+_{\gamma_2}(u_{\gamma_2}) = \kappa$ (by (141)), it follows that $v^+_{\gamma_2}(u_{\gamma_1}) < \kappa = v^+_{\gamma_1}(u_{\gamma_1})$. Aiming for a contradiction, suppose that there exists some $\bar{w} \in [w_1, u_{\gamma_1})$ such that $v^+_{\gamma_2}(\bar{w}) \geq v^+_{\gamma_1}(\bar{w})$. It follows that

$$\hat{w} \;\coloneqq\; \sup\big\{w_1 \leq w < u_{\gamma_1} \,:\, v^+_{\gamma_2}(w) \geq v^+_{\gamma_1}(w)\big\}$$

is well-defined. By the intermediate value theorem, the continuity of $v^+_{\gamma_1}$ and $v^+_{\gamma_2}$, and the definition of $\hat{w}$, we have $v^+_{\gamma_1}(\hat{w}) = v^+_{\gamma_2}(\hat{w})$ and $v^+_{\gamma_1}(w) > v^+_{\gamma_2}(w)$ for all $w \in (\hat{w}, u_{\gamma_1}]$. From (137), it follows that

$$(v^+_{\gamma_1})'(\hat{w}) - (v^+_{\gamma_2})'(\hat{w}) \;=\; \frac{2}{\sigma^2}\,(\gamma_1 - \gamma_2) \;<\; 0.$$

To complete the proof, observe that for each positive integer n such that $\hat{w} + n^{-1} < u_{\gamma_1}$, we have

$$n\big[\big(v^+_{\gamma_1}(\hat{w} + n^{-1}) - v^+_{\gamma_2}(\hat{w} + n^{-1})\big) - \big(v^+_{\gamma_1}(\hat{w}) - v^+_{\gamma_2}(\hat{w})\big)\big] \;=\; n\,\big(v^+_{\gamma_1}(\hat{w} + n^{-1}) - v^+_{\gamma_2}(\hat{w} + n^{-1})\big) \;>\; 0.$$

By the mean value theorem, for each such integer n, there exists $c_n \in (\hat{w}, \hat{w} + n^{-1})$ such that $(v^+_{\gamma_1})'(c_n) - (v^+_{\gamma_2})'(c_n) > 0$. Since $\lim_{n\to\infty} c_n = \hat{w}$, it follows from the continuity of $(v^+_{\gamma_1})'$ and $(v^+_{\gamma_2})'$ that $(v^+_{\gamma_1})'(\hat{w}) - (v^+_{\gamma_2})'(\hat{w}) = \lim_{n\to\infty}\big[(v^+_{\gamma_1})'(c_n) - (v^+_{\gamma_2})'(c_n)\big] \geq 0$. However, this contradicts $(v^-_{\gamma_2})'(\hat{w}) > (v^-_{\gamma_1})'(\hat{w})$. □

Lemma 20. *The mapping $\gamma \mapsto v_\gamma^+(w_1)$ is strictly decreasing with $v_0^+(w_1) \in (0, \kappa)$.*

Proof. By Lemma 17, it follows that the mapping $\gamma \mapsto v_\gamma^+(w_1)$ is strictly decreasing. To complete the proof, we must show that $v_0^+(w_1) \in (0, \kappa)$. First, by Lemma 8 and (139), we have that $w_1 < u_0$. Moreover, since v_0^+ is strictly increasing on $[w_1, u_0]$ (by Lemma 17) with $v_0^+(u_0) = \kappa$ (by (138)), it follows that $v_0^+(w_1) < \kappa$. To show that $v_0^+(w_1) > 0$, assume for contradiction that $v_0^+(w_1) \leq 0$. It then follows from the continuity of v_0^+ and the intermediate value theorem that there exists some $\bar{w} \in [w_1, u_0)$ such that $v_0^+(\bar{w}) = 0$. From (137), this implies that

$$(v_0^+)'(\bar{w}) \;=\; -\frac{2}{\sigma^2}\,\phi(\bar{w}, 0) \;\leq\; 0,$$

where the inequality follows from Lemma 8 and since $\bar{w} \geq w_1$. However, this contradicts Lemma 17. We conclude that $v_0^+(w_1) > 0$, which completes the proof. □

Lemma 21. *The mapping $\gamma \mapsto v_\gamma^+(w_1)$ is continuous on $[0, \infty)$.*

Proof. Similar to the proof of Lemma 14, we must show that for all $\gamma \geq 0$ and $\epsilon > 0$, there exists $\delta = \delta(\gamma, \epsilon) > 0$ such that for all $\gamma' \in (\gamma - \delta, \gamma + \delta) \cap [0, \infty)$, we have $|v_\gamma^+(w_1) - v_{\gamma'}^+(w_1)| < \epsilon$. The added complexity here is that both the number of breakpoints B_γ and their values $\tau_{\gamma,j}^+$ for $j = 0, 1, \ldots, B_\gamma$ are functions of γ. However, it can be shown that these breakpoints also vary continuously with γ. Thus, the mathematical argument closely mirrors that of Lemma 14 and is therefore omitted. □

B.2.4 Existence and Uniqueness of the Solution to the Bellman Equation

In this section, we use the results from Section B.2.3 to prove the existence and uniqueness of a solution to the Bellman equation (60)–(61). Specifically, we construct a family of candidate solutions by pasting together the solutions to the IVPs (116)–(117) and (137)–(138), derived in Sections B.2.3.1 and B.2.3.2, respectively. We then show that a function from this family uniquely solves the Bellman equation. To that end, for $\gamma \in [0, \gamma_k]$, we define the function $v_\gamma : [l_\gamma, u_\gamma] \to \mathbb{R}$ as follows:

$$v_\gamma(w) \;:=\; \begin{cases} v_\gamma^-(w), & w \in [l_\gamma, w_1], \\ v_\gamma^+(w), & w \in (w_1, u_\gamma], \end{cases} \tag{155}$$

where $v_\gamma^- \in C^1[l_\gamma, w_1]$ is the unique solution to (116)–(117) (see Lemma 15) and $v_\gamma^+ \in C^1[w_1, u_\gamma]$ is the unique solution to (137)–(138) (see Lemma 18). The following result proves that there exists a unique γ for which v_γ is continuously differentiable.

Lemma 22. *There exists a unique $\gamma^\star \in [0, \gamma_\kappa]$ such that $v_{\gamma^\star} \in C^1[l_{\gamma^\star}, u_{\gamma^\star}]$. Furthermore, $\gamma^\star > 0$.*

Proof. First, we show that there exists a unique $\gamma^\star \in [0,\gamma_\kappa]$ such that $v^-_{\gamma^\star}(w_1) = v^+_{\gamma^\star}(w_1)$. Define the function $\tilde{v} : [0,\gamma_\kappa] \to \mathbb{R}$ as follows:

$$\tilde{v}(\gamma) \coloneqq v^+_\gamma(w_1) - v^-_\gamma(w_1), \quad \gamma \in [0,\gamma_\kappa].$$

By Lemmas 13 and 20, it follows that $\tilde{v}$ is strictly decreasing on $[0,\gamma_\kappa]$, with

$$\tilde{v}(0) = v^+_0(w_1) \in (0,\kappa) \quad \text{and} \quad \tilde{v}(\gamma_\kappa) = v^+_{\gamma_\kappa}(w_1) - v^-_{\gamma_\kappa}(w_1) < 0. \tag{156}$$

Moreover, Lemmas 14 and 21 imply $\tilde{v}$ is continuous on $[0,\gamma_\kappa]$. Thus, by the intermediate value theorem, the strictly decreasing nature of $\tilde{v}$, and (156), there exists a unique $\gamma^* \in [0,\gamma_\kappa]$ such that $\tilde{v}(\gamma^\star) = 0$. In particular, noting that $\tilde{v}(0) > 0$ and $\tilde{v}(\gamma^\star) = 0$, it follows immediately that $\gamma^\star > 0$.

To complete the proof, it remains to show that $v_{\gamma^\star} \in C^1[l_{\gamma^\star}, u_{\gamma^\star}]$. By Lemmas 10 and 16, together with (155), we know that $v_{\gamma^\star}$ is continuously differentiable on $[l_{\gamma^\star}, w_1) \cup (w_1, u_{\gamma^\star}]$. Since we have already established that $v^-_{\gamma^\star}(w_1) = v^+_{\gamma^\star}(w_1)$, continuity at w_1 follows immediately. Therefore, it remains to show that the derivative $v'_{\gamma^\star}$ exists and is continuous at w_1. Since $v_{\gamma^\star}$ is continuous at w_1 and differentiable both to the left and right of w_1, it suffices to show that $\lim_{w\nearrow w_1} v'_{\gamma^\star}(w) = \lim_{w\searrow w_1} v'_{\gamma^\star}(w)$. However, by (116) and (137), together with the continuity of $v_{\gamma^\star}$, this reduces to verifying that $\lim_{w\nearrow w_1} \phi(w, v^-_{\gamma^\star}(w)) = \lim_{w\searrow w_1} \phi(w, v^+_{\gamma^\star}(w))$. To that end, note that

$$\lim_{w\nearrow w_1} \phi\big(w, v^-_{\gamma^\star}(w)\big) = 0 = \lim_{w\searrow w_1} \frac{\hat{\alpha}_{k^\star(w,v^+_{\gamma^\star}(w))}\big(v^+_{\gamma^\star}(w)\big)\,(w-w_1)}{m_{k^\star(w,v^+_{\gamma^\star}(w))}} = \lim_{w\searrow w_1} \phi\big(w, v^+_{\gamma^\star}(w)\big),$$

where the first equality follows from Corollary 4 for $(w,y) \in [w_0,w_1] \times (-\infty,\kappa]$ (since $v^-_{\gamma^\star}(w) \le \kappa$ for all $w \in [l_{\gamma^\star}, w_1]$), the second equality since $\hat{\alpha}_{k^\star(w,v^+_{\gamma^\star}(w))}(v^+_{\gamma^\star}(w))/m_{k^\star(w,v^+_{\gamma^\star}(w))}$ is bounded on $[w_1,u_\gamma]$ (since $v^+_{\gamma^\star}(w)$ is bounded on $[w_1,u_\gamma]$), and the third equality from Corollary 4 for $(w,y) \in [w_1,\infty) \times (-\infty,\kappa]$ (since $v^-_{\gamma^\star}(w) \le \kappa$ for all $w \in [w_1, u_{\gamma^\star}]$). We conclude that $v'_{\gamma^\star}$ exists and is continuous at w_1. □

The remaining results now show that there exists a unique solution to the Bellman equation (60)–(61).

Corollary 6. *The tuple $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ is a solution to the Bellman equation (60)–(61). Moreover, $l_{\gamma^\star} < w_0 \le w_1 < u_{\gamma^\star}$ and the solution $v_{\gamma^\star} \in C^1[l_{\gamma^\star}, u_{\gamma^\star}]$ is nonnegative and strictly increasing.*

Proof. From (116)–(118), it follows that $v_{\gamma^\star}(l_{\gamma^\star}) = v^-_{\gamma^\star}(l_{\gamma^\star}) = 0$ and $v'_{\gamma^\star}(l_{\gamma^\star}) = (v^-_{\gamma^\star})'(l_{\gamma^\star}) = 0$. Similarly, from (137)–(139), it follows that $v_{\gamma^\star}(u_{\gamma^\star}) = v^+_{\gamma^\star}(u_{\gamma^\star}) = \kappa$ and $v'_{\gamma^\star}(u_{\gamma^\star}) = (v^+_{\gamma^\star})'(u_{\gamma^\star}) = 0$. Therefore, $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ satisfies (61). Next, from (116) and Lemma 15, it follows that $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v^-_{\gamma^\star})$ satisfies (60) for $w \in [l_{\gamma^\star}, w_1]$. Similarly, from (137) and Lemma 18, it follows that $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v^+_{\gamma^\star})$ satisfies (60) for $w \in [w_1, u_{\gamma^\star}]$. Therefore, $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ satisfies (60) for all $w \in [l_{\gamma^\star}, u_{\gamma^\star}]$. We conclude that the tuple $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ is a solution to the Bellman equation (60)–(61).

Since $\gamma^\star > 0$ by Lemma 22, it follows from (118), (139), and Lemma 8 that $l_{\gamma^\star} < w_0$ and $u_{\gamma^\star} > w_1$.

Finally, by Lemmas 11, 17, and 22, the function $v_{\gamma^\star}$ is continuous, with $v^-_{\gamma^\star}$ strictly increasing on $[l_{\gamma^\star}, w_1]$ and $v^+_{\gamma^\star}$ strictly increasing on $[w_1, u_{\gamma^\star}]$. It follows from (155) that $v_{\gamma^\star}$ is strictly increasing on $[l_{\gamma^\star}, u_{\gamma^\star}]$. In particular, since $v_{\gamma^\star}(l_{\gamma^\star}) = 0$, we conclude that $v_{\gamma^\star}$ is nonnegative. □

Lemma 23. *If (l, u, γ, v) is a solution to the Bellman equation (60)–(61), then $l < w_0$, $u > w_1$, and $\gamma \in [0, \gamma_\kappa]$. Moreover, the endpoints are uniquely determined as $u = u_\gamma$ and $l = l_\gamma$.*

Proof. Let (l, u, γ, v) be a solution to the Bellman equation (60)–(61). Then, by definition, we have that $l < u$. We will establish that $l < w_0$, $u > w_1$, and $0 \le \gamma \le \gamma_\kappa$ via contradiction. First, suppose $\gamma < 0$. Then it follows from (60)–(61) that $\phi(l, 0) = \gamma < 0$, contradicting Lemma 8. Thus, we conclude that $\gamma \ge 0$.

Second, suppose $u \le w_1$. We consider the following two cases: $w_0 \le u \le w_1$ and $u < w_0$. If $w_0 \le u \le w_1$, it follows from Lemma 8 that $\phi(u, \kappa) = 0$. However, from (60)–(61), we also have $\phi(u, \kappa) = \gamma - \mu\kappa + m'H^{-1}m\kappa^2/4 > 0$, which is a contradiction. If instead $u < w_0$, it follows from Lemma 8 that $u < l$ (since $\gamma < \gamma - \mu\kappa + m'H^{-1}m\kappa^2/4$). However, this contradicts the fact that $l < u$. Since both cases yield contradictions, we conclude that $u > w_1$. Having established that $u > w_1$, note that since $w \mapsto \phi(w, \kappa)$ is strictly increasing on $[w_1, \infty)$, and hence injective, it follows that $u = u_\gamma$, where u_γ is given by (139).

Third, suppose $l \ge w_0$. Since $u = u_\gamma$, it follows from (60)–(61) that v satisfies (137)–(138). Therefore, by the uniqueness of the solution $v^+_\gamma \in C^1[w_1, u_\gamma]$ to (137)–(138), it follows that $v(w) = v^+_\gamma(w)$ for $w \in [l \vee w_1, u_\gamma]$. In particular, by Lemma 17, we have that $v'(l \vee w_1) > 0$. We now consider the following two cases: $w_0 \le l \le w_1$ and $l > w_1$. If $w_0 \le l \le w_1$, then it follows from (61) and Corollary 4 that $\gamma = 0$. Substituting $\gamma = 0$ into (60)–(61), we see that v satisfies (116)–(117) with γ and l_γ replaced by 0 and l, respectively. Since the zero function satisfies this IVP, it follows from the uniqueness of the construction in Appendix B.2.3.1 that $v(w) = 0$ for $w \in [l, w_1]$. But, since v is continuously differentiable over $[l, u]$, we have that $v'(w_1) = 0$. However, this contradicts $v'(l \vee w_1) = v'(w_1) > 0$. If instead $l > w_1$, it follows that $v'(l \vee w_1) = v'(l) > 0$, contradicting (61). Since both cases yield contradictions, we conclude that $l < w_0$. Having established that $l < w_0$, note that since $w \mapsto \phi(w, 0)$ is strictly decreasing on $(-\infty, w_0]$, and hence injective, it follows that $l = l_\gamma$, where l_γ is given by (118).

Finally, we show that $\gamma \le \gamma_\kappa$. It suffices to show that $v(w) \le \kappa$ for all $w \in [l, u]$. (If $v(w) \le \kappa$ for all $w \in [l, u]$ but $\gamma > \gamma_\kappa$, then from (60)–(61), (116)–(117), (136), and Lemma 13, it would follow that $v(w_1) > \kappa$, which is a contradiction.) Aiming for a contradiction, suppose that $v(w) > \kappa$ for some $w \in [l, u]$. Since $v \in C^1[l, u]$ and satisfies (61), there must exist some $\hat{w} \in (l, u)$ such that $v(\hat{w}) > \kappa$ and $v'(\hat{w}) < 0$. To arrive at a contradiction, it suffices to show that $v'(\hat{w}) > v'(u)$ (since $v'(u) = 0$ by (61), this would contradict $v'(\hat{w}) < 0$). From (60), together with $v(\hat{w}) > \kappa$ and $v(u) = \kappa$, it is enough to show that $\phi(\hat{w}, v(\hat{w})) \le \phi(u, \kappa)$. Since $v(\hat{w}) > \kappa$, it follows from (66) that $\hat{\alpha}_k(v(\hat{w})) \le \hat{\alpha}_k(y)$ for all $y \in (-\infty, \kappa]$ and $k \in \mathcal{S}$. By (59) and (67)–(68), we then have that $\phi(\hat{w}, v(\hat{w})) \le \phi(\hat{w}, y)$ for all $y \in (-\infty, \kappa]$. Therefore, to complete the proof, it is enough to show that $\phi(\hat{w}, \kappa) \le \phi(u, \kappa)$. To do so, we consider the following two cases: $w_0 \le \hat{w} < u$

and $l < \hat{w} < w_0$. If $w_0 \le \hat{w} < u$, then the desired inequality follows immediately from Lemma 8. If instead $l < \hat{w} < w_0$, then it follows from (60)–(61) that $\phi(l, 0) = \gamma < \phi(u, \kappa)$. Furthermore, by (108) and Corollary 4, we have that $\phi(l, 0) = \phi(l, \kappa)$. It then follows from Lemma 8 that $\phi(\hat{w}, \kappa) < \phi(l, \kappa) < \gamma$, as desired. This completes the proof. □

Corollary 7. *The solution to the Bellman equation (60)–(61) is unique.*

Proof. From Lemma 23, any solution (l, u, γ, v) to the Bellman equation (60)–(61) must satisfy $l = l_\gamma$, $u = u_\gamma$, and $\gamma \in [0, \gamma_\kappa]$. But then, by (60)–(61), v must also satisfy the IVPs (116)–(117) and (137)–(138). By Lemmas 15 and 18, uniqueness implies that $v = v_\gamma^-$ on $[l, w_1]$ and $v = v_\gamma^+$ on $(w_1, u]$. Therefore, by (155), it follows that $v = v_\gamma$. Moreover, since v is continuously differentiable, it follows that $\gamma = \gamma^\star$, where $\gamma^\star$ is given by Lemma 22 (otherwise v would not be continuous at w_1). We conclude that the tuple $(l_{\gamma^\star}, u_{\gamma^\star}, \gamma^\star, v_{\gamma^\star})$ from Corollary 6 is the unique solution to (60)–(61). □

C Proofs of Main Results

Proof of Proposition 1. Fix n and let (λ^n, R^n, T^n) be an arbitrary admissible policy for the nth system. Then, by (14), the cumulative profit process for the nth system is given by

$$V^n(t) := \int_0^t \Pi^n\big(\lambda^n(s)\big)\,ds - \sum_{k\in\mathcal{S}} \int_0^t v_k^n\big(w_k^n(s) - \delta_k^n\big)\,dA_k^n(s) - \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k^n\big[Q_k^n(s)\big]^- ds$$
$$- \sum_{k\in\mathcal{S}_{\mathrm{w}}} d_k^n M_k\Big(\int_0^t \eta_k\big[Q_k^n(s)\big]^+ ds\Big) - \sum_{k\in\mathcal{S}} r_k^n R_k^n(t). \tag{157}$$

Since each of the last four terms on the right-hand side of (157) is nonpositive, it follows that

$$V^n(t) \le \int_0^t \Pi^n\big(\lambda^n(s)\big)\,ds = n\int_0^t \Pi\big(n^{-1}\lambda^n(s)\big)\,ds \quad \text{for all} \quad t \ge 0. \tag{158}$$

where the equality follows from the second equality in (18). Moreover, by (20), we have $n^{-1}\lambda^n(s) = \lambda^\star + n^{-1/2}\zeta(s)$, implying that $n^{-1}\lambda^n(s) \in \mathcal{L}$ for all n sufficiently large. Since $\Pi(\lambda) \le \Pi(\lambda^\star)$ for all $\lambda \in \mathcal{L}$ by Assumption 3, it follows from (158) that

$$V^n(t) \le n\int_0^t \Pi(\lambda^\star)\,ds = n\,\Pi(\lambda^\star)\,t \quad \text{for all} \quad t \ge 0,$$

for n sufficiently large. □

Proof of Proposition 2.[22] Let (ζ, O, Y) be an admissible policy for the BCP (39), with the corresponding state descriptor $Z = \{Z(t) : t \geq 0\}$ given by (31)–(33). We will show that there exists an admissible policy (L, U, z, θ) for the EWF (51) with a long-run average cost less than or equal to that of the policy (ζ, O, Y) for the BCP (39). Define the processes

$$L(t) \coloneqq \sum_{k \in \mathcal{S}^{\mathrm{MTO}} \cup \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} Y_k(t), \quad U(t) \coloneqq \sum_{k \in \mathcal{S}} m_k O_k(t), \quad \text{and} \quad \theta(t) \coloneqq \sum_{k \in \mathcal{S}} m_k \zeta_k(t), \quad t \geq 0. \tag{159}$$

Next, define $z(t, w) \coloneqq Z(t)$ for $t \geq 0$ and $w \in \mathbb{R}$ and $W(t) \coloneqq m' Z(t)$ for $t \geq 0$. It then follows from (31)–(33) and (159) that

$$W(t) = B(t) + \int_0^t \theta(s)\, ds - \int_0^t \sum_{k \in \mathcal{S}_{\mathrm{w}}} m_k \ell_k z_k^+(s, W(s))\, ds + L(t) - U(t), \quad t \geq 0, \tag{160}$$

By (36) and (160), it follows that (47)–(48) hold. Moreover, by (34)–(35) and (38), it follows that (50) and (49) hold. Hence, (L, U, z, θ) is an admissible policy for the EWF (51). Finally, by (43)–(44), observe that

$$c(\theta(t)) \leq \zeta(t) H^{-1} \zeta(t) \quad \text{and} \quad \kappa\, U(t) \leq \sum_{k \in \mathcal{S}} r_k O_k(t) \quad \text{for all} \quad t \geq 0.$$

It then follows from (30) and (51) that (L, U, z, θ) achieves a lower cost for the EWF (51) than (ζ, O, Y) does for the BCP (39).

Conversely, let (L, U, z, θ) be an admissible policy for the EWF (51), with the corresponding state descriptor $W = \{W(t) : t \geq 0\}$ given by (47). We will show that there exists an admissible policy (ζ, O, Y) for the BCP (39) with a long-run average cost equal to that of the policy (L, U, z, θ) for the EWF (51). Define the processes

$$O_k(t) \coloneqq \begin{cases} U(t)/m_k, & k = k^\star, \\ 0, & k \neq k^\star, \end{cases} \quad \text{and} \quad \zeta(t) \coloneqq \frac{H^{-1} m}{m' H^{-1} m}\, \theta(t), \quad t \geq 0. \tag{161}$$

By (43) and Lemma 1, it follows that $m' \zeta(t) = \theta(t)$ for $t \geq 0$. Next, define the state descriptor $Z = \{Z(t) : t \geq 0\}$ as $Z_k(t) \coloneqq z_k(t, W(t))$ for $k \in \mathcal{S}$ and $t \geq 0$. Moreover, define the process $Y = \{Y(t) : t \geq 0\}$ as follows:

$$Y_k(t) \coloneqq m_k \Big(Z_k(t) - X_k(t) - \int_0^t \zeta_k(s)\, ds + \int_0^t \mathbb{1}_{\{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTO}}\}} \ell_k Z_k^+(s)\, ds + O_k(t) \Big), \quad k \in \mathcal{S}^{\mathrm{MTO}}, \tag{162}$$

$$Y_k(t) \coloneqq m_k \Big(Z_k(t) - X_k(t) - \int_0^t \zeta_k(s)\, ds + \int_0^t \ell_k Z_k^+(s)\, ds + O_k(t) + \sum_{j \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}(k)} \mu_j Y_j(t) \Big), \quad k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}, \tag{163}$$

$$Y_k(t) \coloneqq m_k \Big(Z_k(t) - X_k(t) - \int_0^t \zeta_k(s)\, ds + O_k(t) \Big), \quad k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}, \tag{164}$$

[22] This proof requires the technical subtly that $B(t) = \sum_{k \in \mathcal{S}} m_k X_k(t)$ almost surely for $t \geq 0$, where X_k for $k \in \mathcal{S}$ and B are the Brownian motions from Sections 4 and 5, respectively. Since B and $\sum_{k \in \mathcal{S}} m_k X_k$ are identically distributed, this equality can be achieved by potentially expanding the probability space on which they are defined; see Skorokhod's representation theorem. For a rigorous treatment of this issue, see Harrison and Williams (2005).

It then readily follows from (50), (48), and (162)–(164) that (31)–(33), (36), and (38) hold. Moreover, by (47)–(48) and (161)–(164), we have that

$$L(t) \;=\; \sum_{k\in\mathcal{S}^{\mathrm{MTO}}\cup\mathcal{S}^{\mathrm{MTS}}_{\mathrm{w}}} Y_k(t), \quad t\ge 0. \tag{165}$$

Therefore, by (49), (161), and (165), it follows that (34)–(35) hold. Hence, (ζ, O, Y) is an admissible policy for the BCP (39). Finally, by (30), (43), (51), (161), and Lemma 1, the policy (ζ, O, Y) achieves the same cost for the BCP (39) as (L, U, z, θ) does for the EWF (51). □

Proof of Proposition 3. Fix a barrier policy (L, U, z, θ) with a lower barrier at l and an upper barrier at u, and assume that $\gamma\in\mathbb{R}$ and $f\in C^2[l,u]$ jointly satisfy (53)–(54). From (49) and Harrison (2013, Appendix B.2), the control processes L and U are of finite variation almost surely. Using this fact and applying Itô's Lemma to the workload process in (41) yields

$$f(W(t)) \,-\, f(W(0)) \;=\; \int_0^t f'(W(s))\,dW(s) \,+\, \int_0^t \frac{1}{2}\sigma^2 f''(W(s))\,ds, \quad t\ge 0. \tag{166}$$

Then, by (41), (52), and (166), it follows that

$$\begin{aligned} f(W(t)) \,-\, f(W(0)) \;=\; &\int_0^t \sigma f'(W(s))\,d(B(s)-\mu s) \,+\, \int_0^t \Gamma_{z,\theta} f(W(s))\,ds \\ &+ \int_0^t f'(W(s))\,dL(s) \,-\, \int_0^t f'(W(s))\,dU(s), \quad t\ge 0. \end{aligned} \tag{167}$$

Recall that by Definition 1, the control processes L and U increase only when $W(t)=l$ and $W(t)=u$, respectively. Thus, it follows from (54) that

$$\int_0^t f'(W(s))\,dL(s) \;=\; f'(l)L(t) \;=\; 0 \quad \text{and} \quad \int_0^t f'(W(s))\,dU(s) \;=\; f'(u)U(t) \;=\; \kappa U(t), \quad t\ge 0. \tag{168}$$

Substituting (168) into (167) and taking expectations of both sides yields

$$\begin{aligned} \mathbb{E}\big[f(W(t)) - f(W(0))\big] \;&=\; \mathbb{E}\Big[\int_0^t \sigma f'(W(s))\,d(B(s)-\mu s)\Big] \,+\, \mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f(W(s))\,ds\Big] \,-\, \mathbb{E}\Big[\int_0^t \kappa\,dU(s)\Big] \\ &=\; \mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f(W(s))\,ds\Big] \,-\, \mathbb{E}\Big[\int_0^t \kappa\,dU(s)\Big], \end{aligned} \tag{169}$$

where the second equality holds by Harrison (2013, Proposition 4.7) since

$$\mathbb{E}\Big[\int_0^t \big[f'(W(s))\big]^2\,ds\Big] \;\le\; \int_0^t \sup_{w\in[l,u]} \big[f'(w)\big]^2\,ds \;<\; \infty,$$

which is finite since f' is a continuous function over the compact set $[l,u]$. Dividing both sides of (169) by t, using (53), and rearranging terms gives

$$\frac{1}{t}\,\mathbb{E}\big[f(W(t)) - f(W(0))\big] \,+\, \frac{1}{t}\,\mathbb{E}\Big[\int_0^t c\big(\theta(W(s))\big)\,ds \,+\, \sum_{k\in\mathcal{S}} \int_0^t v_k\big(z_k(W(s)) - \lambda_k^\star \delta_k\big)\,ds$$

$$+ \sum_{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k z_k^-(W(s))\, ds + \sum_{k \in \mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k z_k^+(W(s))\, ds + \kappa U(t) \Bigg] = \gamma. \tag{170}$$

Finally, we have $\lim_{t\to\infty} t^{-1}\, \mathbb{E}[f(W(t)) - f(W(0))] = 0$, since f is continuous and $W(t) \in [l, u]$ for all $t \geq 0$ by Definition 1. Using this fact and taking the limit as $t \to \infty$ in (170) yields the desired result. □

Proof of Theorem 1. Theorem 1 follows directly from Corollaries 6 and 7 in Appendix B.2.4. Indeed, Appendix B.2.4 provides an explicit characterization of the solution to the Bellman equation (60)–(61). This is accomplished by establishing several auxiliary lemmas in Appendix B.2.3 on two related initial value problems. The solution to the Bellman equation is ultimately constructed by smoothly pasting together the solutions to these initial value problems.

Proof of Corollary 1. Let $(l^\star, u^\star, \gamma^\star, v^\star)$ be the unique solution to the Bellman equation (60)–(61) as guaranteed by Theorem 1. By the fundamental theorem of calculus and the definition of $f^\star \in C^2[l^\star, u^\star]$ in (63), we have that

$$(f^\star)'(w) = v^\star(w) \quad \text{and} \quad (f^\star)''(w) = (v^\star)'(w) \quad \text{for all} \quad w \in [l^\star, u^\star].$$

Since $(l^\star, u^\star, \gamma^\star, v^\star)$ uniquely satisfies (60), by replacing $(f^\star)'$ with $v^\star$ and $(f^\star)''$ with $(v^\star)'$, it immediately follows that $(l^\star, u^\star, \gamma^\star, f^\star)$ uniquely satisfies (56), up to an additive constant in $f^\star$. Furthermore, from the boundary conditions (61), we have that

$$(f^\star)'(l^\star) = v^\star(l^\star) = 0, \quad (f^\star)''(l^\star) = (v^\star)'(l^\star) = 0, \quad (f^\star)''(u^\star) = (v^\star)'(u^\star) = 0, \quad (f^\star)'(u^\star) = v^\star(u^\star) = \kappa,$$

which immediately implies that $(l^\star, u^\star, \gamma^\star, f^\star)$ satisfies (57). We conclude that $(l^\star, u^\star, \gamma^\star, f^\star)$ is the unique solution to the Bellman equation (56)–(57), up to an additive constant in $f^\star$. □

Proof of Theorem 2. Let $(l^\star, u^\star, \gamma^\star, v^\star)$ be the unique solution to the Bellman equation (60)–(61) guaranteed by Theorem 1, and let $f^\star \in C^2[l^\star, u^\star]$ be the function defined in (63). Consider the barrier policy $(L^\star, U^\star, z^\star, \theta^\star)$ with a lower barrier at $l^\star$, an upper barrier at $u^\star$, effective drift rate function $\theta^\star$ defined by (64), and workload configuration function $z^\star$ defined by (65). By (58), (64)–(65), and Corollary 1, it follows that $f^\star$ satisfies (53)–(54) with $z = z^\star$, $\theta = \theta^\star$, and $\gamma = \gamma^\star$. Therefore, by Proposition 3, the barrier policy $(L^\star, U^\star, z^\star, \theta^\star)$ achieves a long-run average expected cost of $\gamma^\star$ to the EWF (51). To complete the proof, it remains to show that $\gamma^\star$ is a lower bound on the long-run average expected cost of any admissible policy (L, U, z, θ) to the EWF (51). The remainder of the proof consists of two parts. In the first part, we extend the solution of the Bellman equation to the entire real line and establish an inequality involving the constant $\gamma^\star$. In the second part, we use the estimate from the first part, along with Itô's Lemma, to prove that no admissible policy can achieve a long-run average expected cost lower than $\gamma^\star$.

We begin the first part of the proof by extending the solution of the Bellman equation to the entire real line. (This is necessary because an arbitrary admissible policy may push the workload process beyond the lower barrier $l^\star$ or the upper barrier $u^\star$.) To that end, we extend $f^\star$ to the entire real line as follows:

$$f^\star(w) := \begin{cases} 0, & w \in (-\infty, l^\star), \\ \displaystyle\int_{l^\star}^{w} v^\star(x)\,dx, & w \in [l^\star, u^\star], \\ \displaystyle\int_{l^\star}^{u^\star} v^\star(x)\,dx + \kappa(w - u^\star), & w \in (u^\star, \infty). \end{cases} \tag{171}$$

We now verify some properties of this extended function. First, since $v^\star \in C^1[l^\star, u^\star]$ with $v^\star(l^\star) = 0$ and $v^\star(u^\star) = \kappa$, it follows from (171) that $f^\star \in C^2(\mathbb{R})$. Second, we have $(f^\star)' \equiv 0$ on $(-\infty, l^\star]$, $(f^\star)' \equiv v$ on $[l^\star, u^\star]$, and $(f^\star)' \equiv \kappa$ on $[u^\star, \infty)$. Therefore, since $v^\star$ is strictly increasing on $[l^\star, u^\star]$ with $v^\star(l^\star) = 0$ and $v^\star(u^\star) = \kappa$ (by Theorem 1), it follows that $0 \le (f^\star)'(w) \le \kappa$ for all $w \in \mathbb{R}$. Next, we claim that

$$\min_{z \in \mathcal{A}(w),\, x \in \mathbb{R}} \left\{ \frac{1}{2}\sigma^2 (f^\star)''(w) + (\mu + x)(f^\star)'(w) + c(x) + \varphi\big(z, (f^\star)'(w)\big) \right\} \ge \gamma^\star, \quad w \in \mathbb{R}. \tag{172}$$

We consider the sets $[l^\star, u^\star]$ and $\mathbb{R} \setminus [l^\star, u^\star]$ separately. On the one hand, for $w \in [l^\star, u^\star]$, (172) holds with equality since $(l^\star, u^\star, \gamma^\star, f^\star)$ satisfies (56)–(57) by Corollary 1. On the other hand, for $w \in \mathbb{R} \setminus [l^\star, u^\star]$, it follows from Lemma 8 and the properties of the extended function $f^\star$ that (172) holds with a strict inequality.[23] This completes the first part of the proof.

The second part of the proof uses the results from the first part to show that our proposed policy is optimal for the workload formulation. Specifically, we show that no admissible policy can achieve an objective lower than $\gamma^\star$. To that end, let (L, U, z, θ) be an arbitrary admissible policy to the EWF (51), and let W be the resulting workload process defined in (47). By applying Itô's Lemma, for $t \ge 0$ we have that

$$\mathbb{E}\Big[f^\star(W(t)) - f^\star(W(0))\Big] = \mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f^\star(W(s))\,ds - \int_0^t (f^\star)'(W(s))\,d(L(s) + U(s))\Big], \tag{173}$$

where the differential operator $\Gamma_{z,\theta}$ is given by (52). Furthermore, by (172), we have that

$$\frac{1}{2}\sigma^2 (f^\star)''\big(W(t)\big) + \big(\mu + \theta(W(t))\,(f^\star)'(W(t)\big) + c\big(\theta(W(t))\big) + \psi\big(z(W(t)), (f^\star)'(W(t))\big) \ge \gamma^\star.$$

Integrating both sides of this inequality over $[0, t]$, taking expectations, and rearranging terms yields

$$\mathbb{E}\Big[\int_0^t \Gamma_{z,\theta} f^\star(W(s))\,ds\Big] \ge \gamma^\star t - \mathbb{E}\Big[\int_0^t c\big(\theta(W(s))\big)\,ds + \sum_{k \in \mathcal{S}} \int_0^t \upsilon_k\big(z_k(W(s)) - \lambda_k^\star \delta_k\big)\,ds$$
$$+ \sum_{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k z_k^-(W(s))\,ds + \sum_{k \in \mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k z_k^+(W(s))\,ds\Big].$$

[23] In particular, (171) implies that $(f^\star)'(w) = (f^\star)'(l^\star) = 0$ and $(f^\star)''(w) = (f^\star)''(l^\star) = 0$ for all $w \in (-\infty, l^\star)$. Similarly, $(f^\star)'(w) = (f^\star)'(u^\star) = \kappa$ and $(f^\star)''(w) = (f^\star)''(u^\star) = 0$ for all $w \in (u^\star, \infty)$. Then, by Lemma 8, observe that $\phi(w, (f^\star)'(w)) = \phi(w, 0) > \phi(l^\star, 0)$ for all $w \in (-\infty, l^\star)$, and $\phi(w, (f^\star)'(w)) = \phi(w, \kappa) > \phi(u^\star, \kappa)$ for all $w \in (u^\star, \infty)$. Combining these inequalities with the fact that (172) holds with equality at $l^\star$ and $u^\star$ establishes that (172) holds with a strict equality for all $w \in \mathbb{R} \setminus [l^\star, u^\star]$.

Moreover, since L and U are nondecreasing (see (49)) and $0 \le (f^\star)'(w) \le \kappa$ for all $w \in \mathbb{R}$, we have that

$$\mathbb{E}\Big[\int_0^t (f^\star)'(W(s))\, dL(s)\Big] \;\ge\; 0 \quad \text{and} \quad \mathbb{E}\Big[\int_0^t (f^\star)'(W(s))\, dU(s)\Big] \;\le\; \mathbb{E}\Big[\kappa U(t)\Big].$$

Substituting the two preceding displays into (173) then yields

$$\mathbb{E}\Big[f^\star(W(t)) - f^\star(W(0))\Big] \;\ge\; \gamma^\star t \;-\; \mathbb{E}\Big[\kappa U(t) \;+\; \int_0^t c\big(\theta(W(s))\big)\, ds \;+\; \sum_{k \in \mathcal{S}} \int_0^t \upsilon_k\big(z_k(W(s)) - \lambda_k^\star \delta_k\big)\, ds$$
$$+ \sum_{k \in \mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \int_0^t h_k\, z_k^-(W(s))\, ds \;+ \sum_{k \in \mathcal{S}_{\mathrm{w}}} \int_0^t d_k \ell_k\, z_k^+(W(s))\, ds\Big]. \tag{174}$$

Finally, since $(f^\star)'$ is bounded, we have that

$$\limsup_{t \to \infty} \frac{\mathbb{E}\big[f^\star(W(t)) - f^\star(W(0))\big]}{t} \;\le\; \limsup_{t \to \infty} \frac{\sup_{w \in \mathbb{R}} (f^\star)'(w) \cdot \mathbb{E}\big[|W(t) - W(0)|\big]}{t} \;=\; 0,$$

where the inequality follows from Jensen's inequality and the mean value theorem, and the equality follows from (50). The desired result then follows by dividing both sides of (174) by t, taking the lim sup as $t \to \infty$, and applying the limiting bound established above on the left-hand side. This completes the second part of the proof. □

Proof of Corollary 2. By Theorem 2, there exists an optimal policy $(L^\star, U^\star, z^\star, \theta^\star)$ to the EWF (51) with a long-run average expected cost of $\gamma^\star$. Therefore, by Proposition 2, there exists an admissible policy $(\zeta^\star, O^\star, Y^\star)$ to the BCP (39) with a long-run average expected cost of $\gamma^\star$. Conversely, by Proposition 2, for every admissible policy (ζ, O, Y) to the BCP (39), there exists an admissible policy (L, U, z, θ) to the EWF (51) with a long-run average expected cost that is less than or equal to that of (ζ, O, Y) to the BCP (39). Hence, $(\zeta^\star, O^\star, Y^\star)$ is an optimal solution to the BCP (39) with a long-run average expected cost of $\gamma^\star$. □

D Supplementary Material for the Simulation Study

D.1 Semi-Markov Decision Process Formulation

This section describes a semi-Markov decision process (SMDP) formulation of the joint dynamic pricing, scheduling, and rejection control problem introduced in Section 3; for a detailed review of the theory of Markov decision processes, see Bertsekas (2012) and Puterman (2014). In this formulation, we assume that production times are exponentially distributed and replace the sojourn times $w_k(t)$ in the objective function (14) by $Q_k(t)/\lambda_k^\star$ for $k \in \mathcal{S}$ and $t \ge 0$. This formulation is used in Section 8 as a benchmark to evaluate the performance of our proposed control policy from Section 7. Under this SMDP model, control decisions—namely, pricing, scheduling, and rejection decisions—are updated only when an order arrives, a product is produced, or a customer abandons. We refer to these moments in time as decision epochs. Due to the

Markovian nature of the system, the time between consecutive decision epochs is exponentially distributed. We now describe the main components of the SMDP formulation: the state space, action space, transition probabilities, and rewards.

State Space. We denote the system state by $q(t) = (q_k(t))$ for $t \geq 0$, where $q_k(t)$ denotes the number of outstanding orders of product k at time t. The state space, denoted by $\mathcal{Q}$, is as follows:

$$\mathcal{Q} := \big\{(q_k) : q_k \in \mathbb{Z} \text{ for } k \in \mathcal{S},\ 0 \leq q_k \leq M_k \text{ for } k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{o}}^{\text{MTS}}, -M_k \leq q_k \leq M_k \text{ for } k \in \mathcal{S}_{\text{w}}^{\text{MTS}}\big\}.$$

Observe that for computational feasibility, we truncate the queue length of product k for $k \in \mathcal{S}$ at $\pm M_k$, where $M_k \in \mathbb{N}$ is a tuning parameter. As detailed in Appendix D.2, we use policy iteration to numerically solve the SMDP. Due to the high memory requirement of the policy evaluation step, we can only solve the SMDP with up to about 650 million states. In the four-product example discussed in Section 8, we use $M_k = 75$ for $k \in \mathcal{S}$. This appears to contain the likely states.

Actions. At each decision epoch, the system manager makes a decision consisting of four components (p, s, u, o), where p is the pricing decision, s is the production decision, u is the reallocation decision, and o is the order rejection decision.

- **Pricing Decisions**: The system manager chooses a price vector $p = (p_k)$, where p_k denotes the price of product $k \in \mathcal{S}$. The set of admissible price vectors is denoted by $\bar{\mathcal{P}} = \mathbb{R}_+^K$.
- **Production Decisions**: The system manager chooses which product to produce, if any. The production decision is denoted by $s \in \{0\} \cup \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$, where $s = 0$ denotes the decision to idle the server, and $s = k$ denotes the decision to produce product $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}$.
- **Reallocation Decisions:** The system manager chooses whether to reallocate (or transfer) finished goods inventory from walk-in MTS class $w(k)$ to online MTS class k. The reallocation decision for product $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$ is denoted by $u_k \in \mathbb{Z}_+ := \{0, 1, 2 \ldots\}$, where $u_k = l$ denotes the decision to reallocate l units of finished goods inventory from class $w(k) \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ to class k.
- **Order Rejection Decisions**: The system manager chooses whether to reject (i.e., deny entry to) the next order of product k. The rejection decision for product $k \in \mathcal{S}$ is denoted by $o_k \in \{0, 1\}$, where $o_k = 1$ indicates rejecting the order and $o_k = 0$ indicates accepting the order.

Given the description of the actions above, the action space $\mathcal{A}$ is given as follows:

$$\mathcal{A} := \big\{(p, s, u, o)\ :\ p \in \bar{\mathcal{P}},\ s \in \{0\} \cup \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}},\ o \in \{0,1\}^K,\ u_k \in \mathbb{Z}_+ \text{ for } k \in \mathcal{S}_{\text{o}}^{\text{MTS}}\big\}.$$

Policies. A policy specifies the decision rule to be used at each decision epoch. To describe the class of policies for the SMDP formulation, we first define the set of feasible actions at each state. Denote by $\mathcal{A}(q)$

the set of feasible actions at state $q \in \mathcal{Q}$, given as follows:

$$\mathcal{A}(q) := \big\{(p,s,u,o) \in \mathcal{A} : s \neq k \text{ if } q_k = 0 \text{ for } k \in \mathcal{S}^{\text{MTO}}, \; s \neq k \text{ if } q_k = -M_k \text{ for } k \in \mathcal{S}_{\text{w}}^{\text{MTS}}, \\ u_k \leq q_k \text{ for } k \in \mathcal{S}_{\text{o}}^{\text{MTS}}, \; \textstyle\sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} u_j \leq q_k^- \text{ for } k \in \mathcal{S}_{\text{w}}^{\text{MTS}}, \\ o_k = 1 \text{ if } q_k = M_k \text{ for } k \in \mathcal{S}\big\}.$$

The requirement that $o_k = 1$ if $q_k = M_k$ for $k \in \mathcal{S}$ ensures that incoming orders for product k are rejected when the queue length of product k is equal to its maximum allowable value M_k. The requirements that $s \neq k$ if $q_k = 0$ for $k \in \mathcal{S}^{\text{MTO}}$ and $s \neq k$ if $q_k = -M_k$ for $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ ensure that production does not occur when the queue length of product k is equal to its minimum allowable value. Finally, the requirements that $u_k \leq q_k$ for $k \in \mathcal{S}_{\text{o}}^{\text{MTS}}$ and $\sum_{j \in \mathcal{S}_{\text{o}}^{\text{MTS}}(k)} u_j \leq q_k^-$ for $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ ensure that the system manager does not reallocate more finished goods inventory to the online classes than necessary, and does not reallocate more from walk-in MTS class $k \in \mathcal{S}_{\text{w}}^{\text{MTS}}$ than its available finished goods inventory, respectively.

A policy is a mapping $\pi : \mathcal{Q} \to \mathcal{A}$ such that $\pi(q) \in \mathcal{A}(q)$ for $q \in \mathcal{Q}$. That is, we restrict attention to stationary Markov policies. The system manager's problem is to choose a policy π that maximizes the long-run average profit over an infinite horizon. For ease of notation, given an action $a \in \mathcal{A}$, we let $p_k(a)$, $\Lambda_k(a)$, $s(a)$, $u_k(a)$, and $o_k(a)$ denote the pricing decision, demand rate, production decision, reallocation decision, and order rejection decision for product k, respectively.

Uniformization. As is standard in the dynamic programming literature, we apply the uniformization technique to write the Bellman equation. To that end, let

$$\Psi := \sup_{p \in \bar{\mathcal{P}}} \sum_{k \in \mathcal{S}} \Lambda_k(p) + \max\{\mu_k : k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}_{\text{w}}^{\text{MTS}}\} + \sum_{k \in \mathcal{S}_{\text{w}}} \ell_k M_k < \infty,$$

which serves as an upper bound on the transition rates. In the uniformized SMDP, the time between two consecutive events is exponentially distributed with rate Ψ. After each event, an action $a \in \mathcal{A}(q)$ is chosen, where $q \in \mathcal{Q}$ denotes the state immediately following the event. The action is not updated until the subsequent event. Under action $a \in \mathcal{A}(q)$, the next event corresponds to the arrival of an order for product $k \in \mathcal{S}$ with probability $\Lambda_k(a)/\Psi$, the production of product $s(a)$ (provided $s(a) \neq 0$) with probability $\mu_{s(a)}/\Psi$, the abandonment of a customer from class $k \in \mathcal{S}_{\text{w}}$ with probability $\ell_k q_k^+/\Psi$, and a fictitious transition otherwise, in which case the system state remains the unchanged.

Transition Probabilities. In the uniformized SMDP, the probability of transitioning from state $q \in \mathcal{Q}$ to state $q' \in \mathcal{Q}$ at the next event under action $a \in \mathcal{A}(q)$ is given as follows:

$$P_{qq'}(a) := \begin{cases} \Lambda_k(a)\mathbb{1}_{\{o_k(a)=0\}}/\Psi, & \text{if } q' = q^a + e_k,\ k \in \mathcal{S}, \\ \sum_{k\in\mathcal{S}} \Lambda_k(a)\mathbb{1}_{\{o_k(a)=1\}}/\Psi, & \text{if } q' = q^a,\ q^a \neq q, \\ (\mu_k \mathbb{1}_{\{s(a)=k\}} + \ell_k q_k^+)/\Psi, & \text{if } q' = q^a - e_k,\ k \in \mathcal{S}_{\mathrm{w}}, \\ \mu_k \mathbb{1}_{\{s(a)=k\}}/\Psi, & \text{if } q' = q^a - e_k,\ k \in \mathcal{S}_{\mathrm{o}}^{\mathrm{MTO}}, \\ 1 - \sum_{\tilde{q}\neq q} P_{q\tilde{q}}(a), & \text{if } q' = q, \\ 0, & \text{otherwise,} \end{cases} \tag{175}$$

where $e_k \in \mathbb{R}^K$ for $k \in \mathcal{S}$ denotes the standard unit basis vector whose kth component equals one and all other components equal zero, $q^a := q + \sum_{k\in\mathcal{S}_{\mathrm{o}}^{\mathrm{MTS}}} u_k(a)(e_{w(k)} - e_k)$ denotes the intermediate state in which reallocation decisions are implemented, and $\mu_0 := 0$. Note that the reallocation decisions are taken and implemented (i.e., products are reallocated) at the decision epochs, namely when a customer order arrives, a product is produced, or a walk-in customer abandons. Although the system state may change after a reallocation decision is implemented, the system manager is not permitted to take another action (e.g., reallocate additional products) until the next decision epoch.

Rewards. The (one-stage) expected reward, i.e., the expected reward earned until the next event, at state $q \in \mathcal{Q}$ under action $a \in \mathcal{A}(q)$ is given by

$$\begin{aligned} R(q,a) := \sum_{k\in\mathcal{S}} \frac{\Lambda_k(a)}{\Psi}\Big(p_k(a) - (\gamma_k + r_k)\mathbb{1}_{\{o_k(a)=1\}}\Big) - \sum_{k\in\mathcal{S}} \frac{\Lambda_k(a)}{\Psi}\,\upsilon_k(q_k^a/\lambda_k^\star - \delta_k)\mathbb{1}_{\{o_k(a)=0\}} \\ - \frac{\mu_{s(a)}}{\Psi}\,\gamma_{s(a)}\mathbb{1}_{\{s(a)\neq 0\}} - \sum_{k\in\mathcal{S}_{\mathrm{w}}} \frac{\ell_k[q_k^a]^+}{\Psi}(\gamma_k + d_k) - \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \frac{[q_k^a]^-}{\Psi} h_k. \end{aligned} \tag{176}$$

Here, the first term denotes the expected revenue, the second term serves as a surrogate for the earliness and tardiness costs, the third term captures the expected production cost, the fourth term accounts for the expected abandonment cost, and the last term represents the expected holding cost.

Bellman Equation. Next, we introduce the Bellman equation for long-run average cost dynamic programming problems, which characterizes an optimal policy:

$$\gamma + f(q) = \sup_{a\in\mathcal{A}(q)} \Big\{R(q,a) + \sum_{q'\in\mathcal{Q}} P_{qq'}(a)f(q')\Big\}, \quad q \in \mathcal{Q}.$$

Here, $\gamma \in \mathbb{R}$ is often interpreted as a guess at the optimal long-run average expected profit. The function $f : \mathcal{Q} \to \mathbb{R}$ is often called a relative value function in average cost dynamic programming. It is easy to see that the relative value function can only be determined up to an additive constant, even if γ is treated as a known constant. Therefore, we set $f(e_0) = 0$, where $e_0 = (0, \dots, 0)$.

D.2 Solving the Semi-Markov Decision Process Formulation

This section describes the policy iteration algorithm used to solve the Bellman equation for the SMDP formulation. To initialize the algorithm, we set $\gamma_0 := 0$ and consider an initial policy π_0 that makes the following pricing, rejection, production, and reallocation decisions. First, it uses the price vector $\Lambda^{-1}(\lambda^\star)$ for all products. Second, it accepts all product $k \in \mathcal{S}$ orders except when $q_k = M_k$, in which case the order is rejected. Third, it produces the product $k \in \mathcal{S}^{\text{MTO}} \cup \mathcal{S}^{\text{MTS}}_{\text{w}}$ with the largest backlog among the products that have a backlog; if no product has a backlog, no production occurs. Finally, for each $k \in \mathcal{S}^{\text{MTS}}_{\text{o}}$, it reallocates $[q_k - \lambda^\star \delta_k]^+$ units of class $w(k)$ inventory to class k. If the available walk-in MTS inventory is insufficient to satisfy all reallocation decisions, the system manager prioritizes the online MTS products in ascending order of their index of their product class. Starting with the initial policy π_0, we iteratively derive policies π_i for $i \in \mathbb{N}$ as follows:

Policy Evaluation. Given the policy $\pi_{i-1} : \mathcal{Q} \to \mathcal{A}$, we solve for the function $f_{i-1} : \mathcal{Q} \to \mathbb{R}$ and the scalar $\gamma_{i-1} \in \mathbb{R}$ that satisfy

$$\gamma_{i-1} \,+\, f_{i-1}(q) \,=\, R(q, \pi_{i-1}(q)) + \sum_{q' \in \mathcal{Q}} P_{qq'}(\pi_{i-1}(q))\, f_{i-1}(q'), \quad q \in \mathcal{Q}, \tag{177}$$

along with the condition $f_{i-1}(e_0) = 0$. Equation (177) corresponds to a sparse system of linear equations, which we solve using the BiCGSTAB iterative sparse solver from the C++ library Eigen.

Policy Improvement. Given the function $f_{i-1} : \mathcal{Q} \to \mathbb{R}$, we define the updated policy $\pi_i : \mathcal{Q} \to \mathcal{A}$ as follows:

$$\pi_i(q) \,:=\, \operatorname*{argmax}_{a \in \mathcal{A}(q)} \big\{ R(q, a) + \sum_{q' \in \mathcal{Q}} P_{qq'}(a) f_{i-1}(q') \big\}, \quad q \in \mathcal{Q}. \tag{178}$$

Termination. The algorithm terminates when the actions under the policy π_i are sufficiently close to those under the previous policy π_{i-1}. To be specific, given an error threshold $\epsilon > 0$, we repeat the policy evaluation and policy improvement steps until, for some $i \in \mathbb{Z}_+$, we have

$$\big|p_k(\pi_i(q)) - p_k(\pi_{i-1}(q))\big| \leq \epsilon, \;\; s(\pi_i(q)) = s(\pi_{i-1}(q)), \;\; u_k(\pi_i(q)) = u_k(\pi_{i-1}(q)), \;\; o_k(\pi_i(q)) = o_k(\pi_{i-1}(q))$$

for all $q \in \mathcal{Q}$ and $k \in \mathcal{S}$.

To facilitate the computation of the updated policy π_i, we show that, given the reallocation decisions, the updated pricing, rejection, and production decisions can be easily characterized. To simplify the analysis, we first write out the definition of the updated policy π_i. Specifically, substituting (175)–(176) into (178) and removing terms independent of the actions yields the following:

$$\pi_i(q) \,=\, \operatorname*{argmax}_{(p,s,u,o) \in \mathcal{A}(q)} \Big\{ \sum_{k \in \mathcal{S}} \frac{\Lambda_k(p)}{\Psi} \Big(p_k - (\gamma_k + r_k) \mathbb{1}_{\{o_k = 1\}} - \upsilon_k (q^a_k / \lambda^\star_k - \delta_k) \mathbb{1}_{\{o_k = 0\}}$$

$$
+ \big(f_{i-1}(q^a + e_k) - f_{i-1}(q)\big)\mathbb{1}_{\{o_k=0\}} + \big(f_{i-1}(q^a) - f_{i-1}(q)\big)\mathbb{1}_{\{o_k=1\}}\Big)
$$

$$
+ \sum_{k\in\mathcal{S}_{\mathrm{w}}} \frac{\ell_k[q_k^a]^+}{\Psi}\big(-(d_k+\gamma_k) + f_{i-1}(q^a - e_k) - f_{i-1}(q)\big) - \sum_{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}} \frac{h_k[q_k^a]^-}{\Psi}
$$

$$
+ \frac{\mu_s}{\Psi}\big(-\gamma_s + f_{i-1}(q^a - e_s) - f_{i-1}(q)\big)\mathbb{1}_{\{s\neq 0\}}\Big\}, \qquad q \in \mathcal{Q}. \tag{179}
$$

The following three lemmas characterize the production, rejection, and pricing decisions given the reallocation decisions. Then, using a simple search over the finite set of feasible reallocation decisions, we obtain the updated reallocation decisions and, consequently, the updated production, rejection, and pricing decisions.

Lemma 24. *Given the reallocation decisions, the updated production decision at state $q \in \mathcal{Q}$ is given by*

$$
s(\pi_i(q)) = \underset{\substack{s\in\{0\}\cup\{k\in\mathcal{S}^{\mathrm{MTO}}:\, q_k>0\}\\ \cup\{k\in\mathcal{S}_{\mathrm{w}}^{\mathrm{MTS}}:\, q_k>-M_k\}}}{\operatorname{argmax}} \frac{\mu_s}{\Psi}\big(-\gamma_s + f_{i-1}(q^a - e_s) - f_{i-1}(q)\big)\mathbb{1}_{\{s\neq 0\}}.
$$

Proof. The production decision s at state $q \in \mathcal{Q}$ only impacts the last term on the right-hand side of (179). Therefore, given the reallocation decisions, the updated production decision can be characterized independently of the other decisions and is the maximizer of the last term on the right-hand side of (179), which yields the result. ☐

Lemma 25. *Given the reallocation decisions, the updated rejection decision for product $k \in \mathcal{S}$ at state $q \in \mathcal{Q}$ is given by*

$$
o_k(\pi_i(q)) = \begin{cases} 0, & \text{if } f_{i-1}(q^a + e_k) - \upsilon_k(q_k^a/\lambda_k^\star - \delta_k) > f_{i-1}(q^a) - (\gamma_k + r_k) \text{ and } q_k < M_k,\\ 1, & \text{otherwise.} \end{cases}
$$

Proof. The rejection decision o_k for product k only impacts the kth term on the right-hand side of (179). Moreover, since $\Lambda_k(p) \geq 0$ for all $p \in \bar{\mathcal{P}}$ and $k \in \mathcal{S}$, the updated prices do not impact the updated rejection decisions. Therefore, the updated rejection decision o_k for product $k \in \mathcal{S}$ at state $q \in \mathcal{Q}$ satisfies

$$
o_k(\pi_i(q)) = \underset{o_k\in\mathcal{O}_k(q)}{\operatorname{argmax}} \big\{ -(\gamma_k + r_k)\mathbb{1}_{\{o_k=1\}} - \upsilon_k(q_k^a/\lambda_k^\star - \delta_k)\mathbb{1}_{\{o_k=0\}}
$$

$$
+ \big(f_{i-1}(q^a + e_k) - f_{i-1}(q^a)\big)\mathbb{1}_{\{o_k=0\}} + \big(f_{i-1}(q^a) - f_{i-1}(q)\big)\mathbb{1}_{\{o_k=1\}}\big\},
$$

where $\mathcal{O}_k(q) = \{1 - \mathbb{1}_{\{q_k<M_k\}}, 1\}$ denotes the set of feasible rejection decisions for product k at state q. The result follows by comparing the two expressions corresponding to $o_k = 0$ and $o_k = 1$, and selecting the value that yields the larger objective. ☐

Lemma 26. *Given the reallocation decisions, the updated pricing decisions at state $q \in \mathcal{Q}$ are given by*

$$
p(\pi_i(q)) = \underset{p\in\bar{\mathcal{P}}}{\operatorname{argmax}} \sum_{k\in\mathcal{S}} \Lambda_k(p)\big(p_k - z_k(q)\big),
$$

where, for $k \in \mathcal{S}$ *and* $q \in \mathcal{Q}$*,*

$$z_k(q) := \big(\gamma_k + r_k - f_{i-1}(q^a)\big)\mathbb{1}_{\{o_k(\pi_i(q))=1\}} + \big(\upsilon_k(q_k^a/\lambda_k^\star - \delta_k) - f_{i-1}(q^a + e_k)\big)\mathbb{1}_{\{o_k(\pi_i(q))=0\}} + f_{i-1}(q).$$

Proof. The pricing decision p at state $q \in \mathcal{Q}$ only impacts the first k terms on the right-hand side of (179). Therefore, given the updated reallocation and rejection decisions, it follows that the updated pricing decision is characterized by the optimization problem in the statement of the lemma. □

Following the approach outlined in Gallego and Wang (2014, Section 2), we reformulate the optimization problem in Lemma 26 as a one-dimensional convex optimization problem. We then solve the reformulated problem using the Brent algorithm (Press, 2007).

In summary, for each feasible reallocation decision at each state, we obtain the updated production, rejection, and pricing decisions as discussed above. Then, we iterate over the feasible reallocation decisions to find the updated reallocation decisions and, consequently, the updated policy; see (179).

D.3 Supplementary Results

This section presents some supplementary results for the simulation study described in Section 8. Table 1 reports the prices used in the base case by the various policies. Tables 2–7 report the long-run average cost of the various policies and their gap from the SMDP benchmark. Specifically, we conduct a sensitivity analysis in which one model parameter is varied at a time, while all others are held fixed at their base case values. The parameters varied include the system load factor, the magnitude of the state costs, the nominal MTS demand, the nominal online demand, and the abandonment rate. The procedure by which each parameter is varied is described in Section 8.3.

Table 1: Comparative evaluation of the prices used in the various policies under the base case. Reported values are the means and the 95% confidence intervals (reported inside the parentheses). The nominal static price vector is $p^\star = (1.0, 1.0, 1.0, 1.0)$.

	Product Class			
	Walk-in MTO	Walk-in MTS	Online MTO	Online MTS
SMDP Policy, Dyn. Prices	1.026 (1.000, 1.096)	1.010 (1.000, 1.039)	1.027 (1.005, 1.108)	1.009 (1.000, 1.039)
BCP Policy, Dyn. Prices	1.025 (1.000, 1.101)	1.011 (1.000, 1.042)	1.025 (1.000, 1.101)	1.011 (1.000, 1.042)
BCP Policy, Online-Only Dyn. Prices	1.033	1.013	1.029 (1.000, 1.133)	1.013 (1.000, 1.056)
SMDP Policy, Static Prices	1.047	1.019	1.047	1.019
BCP Policy, Static Prices	1.049	1.019	1.049	1.019

Table 2: Comparative evaluation of the long-run average cost of the various policies, varying only the load factor. The reported values are sample averages; the standard errors for the long-run average costs and the gaps from the SMDP benchmark are less than or equal to 0.01 and 1.5%, respectively.

	Load Factor							
	0.90	0.95	0.97	1.0	1.03	1.05	1.07	1.1
Long-Run Average Expected Cost								
SMDP Policy, Dyn. Prices	0.14	0.20	0.24	0.32	0.44	0.56	0.70	0.95
BCP Policy, Dyn. Prices	0.14	0.20	0.24	0.32	0.45	0.57	0.71	0.97
BCP Policy, Online-Only Dyn. Prices	0.16	0.23	0.27	0.37	0.52	0.65	0.81	1.10
SMDP Policy, Static Prices	0.16	0.23	0.31	0.44	0.64	0.80	0.99	1.34
BCP Policy, Static Prices	0.16	0.24	0.31	0.45	0.65	0.81	1.00	1.36
Gap from the SMDP Benchmark								
BCP Policy, Dyn. Prices	1.3%	1.1%	1.1%	1.0%	1.7%	1.9%	1.8%	1.8%
BCP Policy, Online-Only Dyn. Prices	14.3%	12.7%	13.4%	15.5%	17.9%	17.5%	16.0%	15.3%
SMDP Policy, Static Prices	9.8%	16.8%	29.2%	39.6%	44.2%	44.2%	42.7%	40.5%
BCP Policy, Static Prices	15.2%	20.6%	30.2%	41.8%	45.7%	44.7%	43.7%	42.9%

Table 3: Comparative evaluation of the long-run average cost of the various policies, varying only the magnitude of the state (earliness, tardiness, and holding) costs. The reported values are sample averages; the standard errors for the long-run average costs and the gaps from the SMDP benchmark are less than or equal to 0.01 and 1.5%, respectively.

	Magnitude of the State Costs							
	0.01	0.03	0.05	0.07	0.1	0.12	0.15	0.2
Long-Run Average Expected Cost								
SMDP Policy, Dyn. Prices	0.12	0.23	0.32	0.40	0.51	0.59	0.69	0.88
BCP Policy, Dyn. Prices	0.12	0.23	0.32	0.41	0.53	0.60	0.71	0.90
BCP Policy, Online-Only Dyn. Prices	0.13	0.26	0.37	0.47	0.59	0.69	0.81	1.01
SMDP Policy, Static Prices	0.17	0.32	0.44	0.55	0.69	0.78	0.89	1.10
BCP Policy, Static Prices	0.17	0.33	0.45	0.55	0.70	0.78	0.90	1.12
Gap from the SMDP Benchmark								
BCP Policy, Dyn. Prices	1.2%	1.7%	1.0%	1.0%	3.1%	2.0%	2.3%	2.2%
BCP Policy, Online-Only Dyn. Prices	12.8%	13.5%	15.5%	15.9%	15.6%	17.7%	17.1%	14.8%
SMDP Policy, Static Prices	41.5%	40.3%	39.6%	36.7%	34.2%	32.4%	28.8%	25.9%
BCP Policy, Static Prices	47.5%	43.5%	41.8%	37.6%	35.5%	33.4%	29.3%	27.4%

Table 4: Comparative evaluation of the long-run average cost of the various policies, varying only the fraction of (nominal) demand that chooses the MTS good. The reported values are sample averages; the standard errors for the long-run average costs and the gaps from the SMDP benchmark are less than or equal to 0.01 and 1.5%, respectively.

	Nominal MTS Demand								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Long-Run Average Expected Cost									
SMDP Policy, Dyn. Prices	0.38	0.36	0.35	0.33	0.32	0.30	0.27	0.25	0.22
BCP Policy, Dyn. Prices	0.38	0.37	0.35	0.34	0.32	0.30	0.28	0.26	0.23
BCP Policy, Online-Only Dyn. Prices	0.43	0.42	0.41	0.39	0.37	0.35	0.33	0.29	0.26
SMDP Policy, Static Prices	0.52	0.51	0.48	0.47	0.44	0.42	0.39	0.36	0.32
BCP Policy, Static Prices	0.52	0.51	0.49	0.47	0.45	0.42	0.39	0.36	0.32
Gap from the SMDP Benchmark									
BCP Policy, Dyn. Prices	0.2%	0.9%	1.5%	0.6%	1.0%	2.6%	2.0%	1.8%	2.5%
BCP Policy, Online-Only Dyn. Prices	15.4%	14.2%	16.2%	15.5%	15.5%	18.5%	20.5%	16.8%	17.4%
SMDP Policy, Static Prices	38.5%	39.4%	38.6%	39.3%	39.6%	41.2%	43.7%	44.5%	46.3%
BCP Policy, Static Prices	38.9%	41.2%	40.9%	39.9%	41.8%	41.9%	44.2%	44.1%	46.8%

Table 5: Comparative evaluation of the long-run average cost of the various policies, varying only the fraction of (nominal) demand that chooses the online channel. The reported values are sample averages; the standard errors for the long-run average costs and the gaps from the SMDP benchmark are less than or equal to 0.01 and 1.5%, respectively.

	Nominal Online Demand								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Long-Run Average Expected Cost									
SMDP Policy, Dyn. Prices	0.51	0.44	0.39	0.35	0.32	0.30	0.29	0.27	0.26
BCP Policy, Dyn. Prices	0.51	0.44	0.39	0.35	0.32	0.31	0.29	0.28	0.27
BCP Policy, Online-Only Dyn. Prices	0.62	0.53	0.46	0.41	0.37	0.34	0.32	0.29	0.27
SMDP Policy, Static Prices	0.63	0.56	0.51	0.47	0.44	0.42	0.40	0.38	0.36
BCP Policy, Static Prices	0.64	0.57	0.51	0.47	0.45	0.43	0.41	0.38	0.36
Gap from the SMDP Benchmark									
BCP Policy, Dyn. Prices	0.8%	0.2%	0.5%	1.1%	1.0%	1.5%	1.6%	2.0%	1.9%
BCP Policy, Online-Only Dyn. Prices	22.9%	21.2%	19.4%	17.6%	15.5%	13.1%	9.8%	5.9%	2.6%
SMDP Policy, Static Prices	25.0%	27.9%	31.5%	35.3%	39.6%	40.1%	40.1%	38.9%	37.3%
BCP Policy, Static Prices	26.6%	29.0%	32.4%	36.3%	41.8%	41.3%	41.1%	39.2%	37.7%

Table 6: Comparative evaluation of the long-run average cost of the various policies, varying only the abandonment rate (of the walk-in channel). The reported values are sample averages; the standard errors for the long-run average costs and the gaps from the SMDP benchmark are less than or equal to 0.01 and 1.5%, respectively.

	Abandonment Rate						
	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Long-Run Average Expected Cost							
SMDP Policy, Dyn. Prices	0.32	0.54	0.72	0.89	1.06	1.21	1.36
BCP Policy, Dyn. Prices	0.32	0.55	0.74	0.92	1.09	1.26	1.43
BCP Policy, Online-Only Dyn. Prices	0.37	0.58	0.77	0.94	1.11	1.28	1.43
SMDP Policy, Static Prices	0.44	0.64	0.82	0.98	1.14	1.29	1.43
BCP Policy, Static Prices	0.45	0.65	0.82	0.99	1.15	1.31	1.47
Gap from the SMDP Benchmark							
BCP Policy, Dyn. Prices	1.0%	1.8%	2.6%	2.9%	3.7%	4.2%	5.2%
BCP Policy, Online-Only Dyn. Prices	15.5%	7.9%	6.7%	5.8%	5.5%	5.8%	5.4%
SMDP Policy, Static Prices	39.6%	19.4%	14.4%	10.4%	7.8%	6.5%	5.5%
BCP Policy, Static Prices	41.8%	19.9%	14.2%	11.0%	9.4%	8.4%	8.2%

Table 7: Comparative evaluation of the long-run average cost of the various policies, varying only the correlation parameter of the nested logit demand model. The reported values are sample averages; the standard errors for the long-run average costs and the gaps from the SMDP benchmark are less than or equal to 0.01 and 1.5%, respectively.

	Nested Logit Correlation Parameter									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Long-Run Average Expected Cost										
SMDP Policy, Dyn. Prices	0.26	0.28	0.29	0.30	0.30	0.31	0.31	0.31	0.32	0.32
BCP Policy, Dyn. Prices	0.27	0.28	0.30	0.31	0.31	0.31	0.32	0.32	0.32	0.32
BCP Policy, Online-Only Dyn. Prices	0.28	0.32	0.34	0.34	0.35	0.36	0.36	0.37	0.37	0.37
SMDP Policy, Static Prices	0.32	0.37	0.40	0.41	0.42	0.42	0.43	0.43	0.44	0.44
BCP Policy, Static Prices	0.32	0.37	0.40	0.41	0.42	0.43	0.43	0.44	0.44	0.45
Gap from the SMDP Benchmark										
BCP Policy, Dyn. Prices	1.5%	1.5%	2.0%	2.9%	2.1%	1.3%	1.5%	1.6%	0.7%	1.0%
BCP Policy, Online-Only Dyn. Prices	7.9%	14.1%	15.4%	14.2%	17.2%	16.7%	16.5%	17.9%	15.4%	15.5%
SMDP Policy, Static Prices	21.1%	31.7%	35.8%	35.9%	37.6%	37.3%	38.9%	39.3%	38.1%	39.6%
BCP Policy, Static Prices	21.4%	32.1%	36.6%	36.7%	38.0%	39.6%	39.7%	39.9%	39.1%	41.8%